Forecast Comparison Tests Under Fat-Tails

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Forecast comparison test

- Suppose that
  - y_t: variable to be forecast
  - F_{1t} & F_{2t}: forecasts of y_t at time $t-1$

- Diebold-Mariano (1995), West (1996), Giacomini-White (2006)

$$H_0 : \mathbb{E}[L(y_t, F_{1t})] = \mathbb{E}[L(y_t, F_{2t})]$$
$$H_1 : \mathbb{E}[L(y_t, F_{1t})] < \mathbb{E}[L(y_t, F_{2t})]$$

where L is the loss function.

- With d_t (loss difference) $d_t = L(y_t, F_{1t}) - L(y_t, F_{2t})$,

$$H_0 : \mathbb{E}[d_t] = 0 \quad \text{vs.} \quad H_1 : \mathbb{E}[d_t] < 0$$

Motivation

- $H_0 : \mathbb{E}[d_t] = 0$

- Inference is typically done based on either

(a) CLT (Normal asymptotics)

(b) Stationary bootstrap (Politis-Romano 1994).

with an assumption that $\mathbb{E}|d_t|^r < \infty$ for some $r > 2$.

This paper: we question validity of moment condition.

1. Cases where the moment cond. may be violated.
2. If so, classical CLT $\rightarrow$ size distortions
3. Subsampling as a robust procedure.

Contribution 1

1. Cases where the moment cond. may be violated (i.e., $\mathbb{E}[d_t^2] = \infty$)

- Depends on the choice of L, nature of y_t and F_{it}, analysis in the paper

- Empirically: find values below or around 2 for various tail index estimators of d_t from variance forecast tests
(Hansen and Lunde (2005), Bollerslev, Patton, Quaedvlieg (BPQ, 2016))

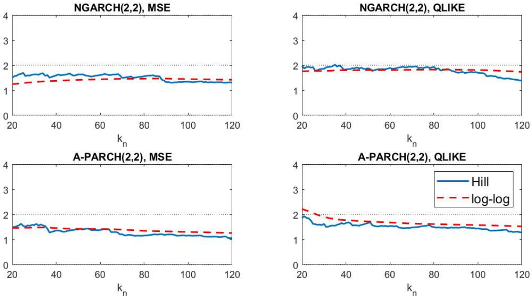
2. Classical CLT leads to size distortions

3. Subsampling

Tail index estimators

Tail index estimators are overall below 2.

$\rightarrow$ No evidence for $\mathbb{E}[d_t^2] < \infty$.



Tail index estimates of loss differences (Hansen and Lunde, 2005.) The panels show the tail index estimates of $d_t = L(RV_t, F_t^j) - L(RV_t, F_t^{GARCH(1,1)})$ with $j \in$ {NGARCH(2,2) , A-PARCH(2,2)}, where F_t^k denotes the variance forecast based on a model k with zero-mean and Normal innovation.

Contribution 2

1. Cases where the moment cond. may be violated

2. If this is the case, classical CLT lead to size distortion

- Assume that d_t is regularly varying time series with index $\alpha \in (1, 2)$.
- Asymptotic distrib. of t-stat under H_0
  - Depends on how thick (α) and how symmetric (p) the tails are
  - Depart from Normality especially when tails are thicker and more asymmetric

    $\rightarrow$ Rejection rate $\neq$ 5% under H_0 with critical value of 1.64

3. Subsampling

Asymptotics under fat-tails

Assume that d_t is strictly stationary, mixing, and there exists $\alpha \in (1, 2)$ such that

$$\mathbb{P}(|d_t| > z) = z^{-\alpha}\ell(z)$$

where ℓ is slowly-varying fn and

$$\frac{\mathbb{P}(d_t > z)}{|d_t| > z} \xrightarrow{z\to\infty} p, \quad \frac{\mathbb{P}(d_t < -z)}{|d_t| > z} \xrightarrow{z\to\infty} 1-p, \quad p \in [0, 1]$$

hold.

Then, under the null hypothesis ($\mathbb{E}(d_t) = 0$),

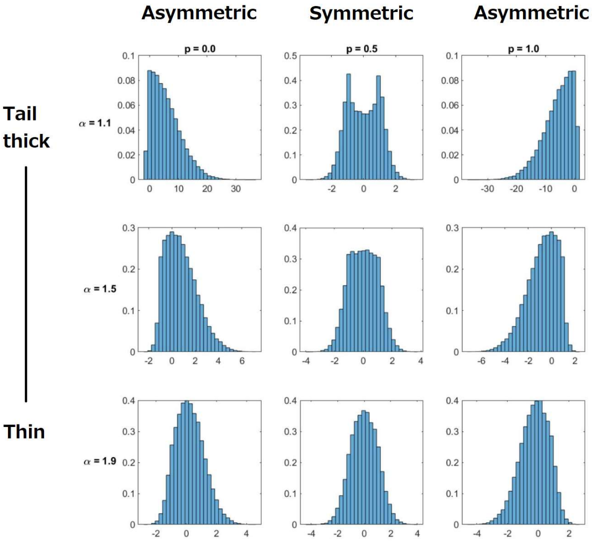
$$\frac{1/T \sum d_t}{\sqrt{\hat{v}ar(d_t)/T}} \xrightarrow{d} M(\alpha, p)$$

- From Davis and Hsing (1995),

$$M(\alpha,p) = \frac{\sum_{j=1}^{\infty}(\delta_j Z_j - (2p-1)\mathbb{E}[Z_j I_{Z_j \in (0,1]}]) - (2p-1)\alpha/(\alpha-1)}{(\sum_{j=1}^{\infty} Z_j^2)^{1/2}}$$

where (δ_j) and (Z_j) are independent and

- $\delta_j \sim iid.$ with $\mathbb{P}(\delta_j = 1) = p$, $\mathbb{P}(\delta_j = -1) = 1 - p$,
- $Z_j = (\sum_{k=1}^{j} E_k)^{-1/\alpha}$ where $E_k \sim iid.exp(1)$.


Asymmetric
Symmetric
Asymmetric
Tail
thick
Thin
p = 0.0
p = 0.5
p = 1.0
α = 1.1
α = 1.5
α = 1.9

Contribution 3

1. $\mathbb{E}[d_t^2] = \infty$ may happen
2. Classical CLT leads to size distortions
3. Subsampling

- Asymptotically valid under fat-tails (Politis, Romano and Wolf, 1999)
- Finite-sample performance depends on the choice of the block sizes.
- Propose a block-size selection rule which work well in simulations

Alternative approach: subsampling

- Block size selection (Romano and Wolf, 2001) requires pre-selected bounds, $[b_{min}, b_{max}]$.

- We propose a formula to obtain b_{min}, b_{max}:

$$\begin{aligned} b_{min} &= C_{min} T^{0.33}, \quad C_{min} = (\beta + 2)\alpha \\ b_{max} &= C_{max} T^{0.66}, \quad C_{max} = 0.5(\beta(2-\alpha) + 2)\alpha^2 \end{aligned}$$

where (α, β) estimated by assuming $d_t \sim \mathbb{S}_\alpha(\sigma, \beta, 0)$

Simulation under the null

DGP under the null: $d_t \sim iid.\mathbb{S}_\alpha(1, \beta, \mu)$ with $\mu = 0$

- $\alpha \in \{1.1, 1.3, 1.6, 1.9\}$
- $\beta \in \{-1, -0.5, 0, 0.5, 1\}$

Each replication: conduct one-sided test of level 5%.

1. Normal asymptotics: reject if $\tau > 1.64$
2. Subsampling w. block size selected according to the formula (w. estimated (α, β))

Simulations on the power property is here.

Rejection Frequency (Nominal rate = 5%)

Normal asymptotics

	T	β = -1	-0.5	0	0.5	1
$\alpha = 1.1$	250	76.30	62.95	2.85	0.00	0.00
	500	76.55	63.00	3.05	0.00	0.00
	1000	77.15	64.10	3.00	0.00	0.00
	2500	77.55	63.35	3.20	0.00	0.00
$\alpha = 1.3$	250	43.75	25.70	3.40	0.20	0.10
	500	43.10	25.20	3.90	0.10	0.00
	1000	46.40	25.70	3.40	0.20	0.05
	2500	44.30	25.35	4.05	0.20	0.00
$\alpha = 1.6$	250	18.50	9.60	3.85	1.45	0.60
	500	17.90	11.05	4.95	1.60	0.40
	1000	17.85	10.35	4.20	1.65	0.75
	2500	17.90	10.15	4.75	1.90	0.70
$\alpha = 1.9$	250	7.10	5.85	4.80	4.30	3.60
	500	8.50	7.25	6.00	5.10	4.20
	1000	6.45	5.55	4.70	3.80	3.25
	2500	6.65	5.70	4.80	3.85	3.25

Subsampling

α	T	β = -1	-0.5	0	0.5	1
1.1	250	5.45	5.30	5.55	1.45	1.60
	500	5.85	5.60	6.25	1.30	1.00
	1000	5.85	5.25	6.00	1.15	1.20
	2500	5.65	6.60	5.50	0.70	0.50
1.3	250	5.30	5.40	5.50	3.75	2.50
	500	5.30	6.40	6.05	3.40	2.25
	1000	5.50	4.95	5.05	2.75	1.95
	2500	5.05	5.35	4.45	2.95	1.60
1.6	250	5.20	4.85	4.90	4.50	4.00
	500	5.10	5.40	4.95	3.55	3.15
	1000	4.10	3.60	3.25	3.00	2.55
	2500	3.90	3.75	3.40	2.95	2.90
1.9	250	5.45	4.85	6.05	5.45	5.55
	500	3.95	3.40	3.35	3.55	2.95
	1000	2.65	2.90	2.50	2.70	2.55
	2500	2.75	2.40	2.35	2.20	2.25

Unit: Percent

Conclusion

- Diebold-Mariano test with $H_0 : \mathbb{E}[d_t] = 0$
- Question the validity of the moment condition, $\mathbb{E}[d_t^2] < \infty$

1. Cases where the moment cond. may be violated
2. If so, classical CLT $\rightarrow$ size distortion
3. Subsampling as a robust approach.