Estimating Growth at Risk with Skewed Stochastic Volatility Models

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1. Motivation

- Adrian et al. (2019) analyze the forecasting density of one-period ahead US GDP growth (gdp_{t+1}) based on current national financial conditions ($nfci_t$) using a semi-parametric approach.
- **Findings:**
  (1) Lower quantiles of the conditional distribution vary over time while upper quantiles remain relatively stable.
  (2) Deterioration of national financial conditions coincides with increases in the interquartile range and skewness.
  (3) Distributions are symmetric in normal times and become left skewed in recessionary periods.
- **Drawbacks of the semi-paramteric approach:**
  - Does not allow for parameter inference
  - No law of motion to obtain multi-step forecasts
- **Aim of this paper: Propose a parametric model to analyze the evolution of the conditional forecast distributions.**

2. Skewed Stochastic Volatility Model

The Skewed Stochastic Volatility Model (SSV) is a non-linear, non-Gaussian state space model with measurement equation

$$y_t = \gamma_0 + \sum_{l=1}^{L} \gamma_l x_{t,l} + \sum_{p=1}^{P} \beta_p y_{t-p} + \varepsilon_t \quad \text{with} \quad \varepsilon_t \sim skew\,\mathcal{N}(0, \sigma_t, \alpha_t)$$

and latent states

$$\ln(\sigma_t) = \delta_{1,0} + \sum_{j=1}^{J_\sigma} \delta_{1,j} x_{t,j} + \sum_{k=1}^{K_\sigma} \beta_{1,k} \ln(\sigma_{t-k}) + \nu_{1,t}$$

$$\alpha_t = \delta_{2,0} + \sum_{j=1}^{J_\alpha} \delta_{2,j} x_{t,j} + \sum_{k=1}^{K_\alpha} \beta_{2,k} \alpha_{t-k} + \nu_{2,t}$$

- $\nu_{1,t}$ and $\nu_{2,t}$ are assumed to be uncorrelated Gaussian White Noise innovations
- Errors in the measurement equation are distributed according to the skewed Normal distribution of Azzalini (2013).

The skew normal distribution has parameters for **location** (μ) and **scale** (σ) plus an additional parameter (α) for the **shape**:

$$skew\,\mathcal{N}(y|\mu,\sigma,\alpha) = \frac{2}{\sqrt{(2\pi)}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \int_{-\infty}^{\alpha\frac{y-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}}\, dt$$

- $nfci_t$ can affect first three moments
- Kurtosis evolves endogenously
- SSV model nests symmetric SV model for $\alpha = 0$

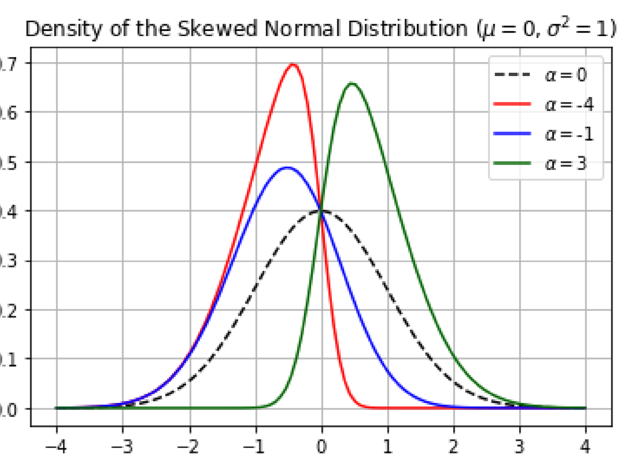

Figure: Skew-Normal Density

3. Estimation Method

Based on the work of Kim et al. (1998), the skewed stochastic volatility models is estimated using a tempered Particle Metropolis Hastings algorithm:

- **Static Model Parameters** $\theta = (\gamma_0, ..., \beta_1, ..., \delta_{1,0}, ..., \beta_{1,1}, ..., \sigma_{\nu,1}, \delta_{2,0}, ..., \beta_{2,1}, ..., \sigma_{\nu,2})$ are estimated using a Metropolis Hastings sampler with stationary distribution:

$$p(\theta|y_{1:T}, s_{1:T}) = \frac{p(y_{1:T}|s_{1:T},\theta)p(s_{1:T}|\theta)p(\theta)}{p(y_{1:T})}$$

- **Time varying model parameters** $s_t = (\ln\sigma_t, \alpha_t)$ are estimated using the tempered particle filter from Herbst and Schorfheide (2019):

$$p(s_t|y_{1:t}) = \frac{p(y_t|s_t)p(s_t|y_{1:t-1})}{\int p(y_t|s_t)p(s_t|y_{1:t-1})ds_t}$$

3.1 Tempered Particle Filter

- Tempered particle filter uses adaptive version of annealed importance sampling to guarantee a targeted inefficiency ratio r^* via a sequence of N_ϕ bridge distributions.
- To increase efficiency of the filter for the SSV, I introduce a modified tempering schedule

$$p_n(y_t|s_{t,i}) = skew\,\mathcal{N}(y_t|\mu_t, \sigma_{t,i}/\phi_n, \phi_n\alpha_{t,i}) \quad \text{with} \quad 0 < \phi_n < 1 \quad \text{and} \quad \lim_{n\to N_\phi} \phi_n = 1.$$

with lower bound on the intitial Inefficiency Ratio

$$\lim_{\phi_0\to 0} Ineff_t(\phi_0) = \frac{\frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{\sigma_{i,t}}\right)^2}{\left(\frac{1}{M}\sum_{i=1}^{M}\frac{1}{\sigma_{i,t}}\right)^2} > 1 \quad \text{and} \quad r^* = \frac{\frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{\sigma_{i,t}}\right)^2}{\left(\frac{1}{M}\sum_{i=1}^{M}\frac{1}{\sigma_{i,t}}\right)^2} + \Delta_r$$

- Tempering the symmetry reduces the number N_ϕ of tempering steps by about 25%.

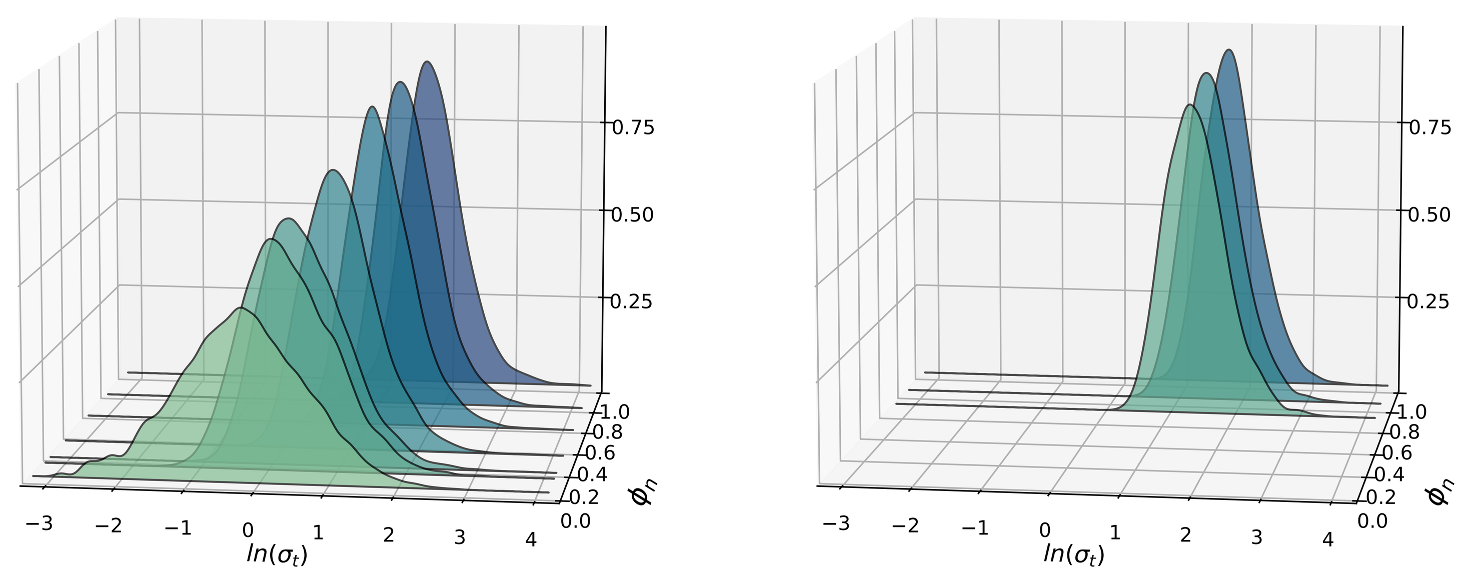

Figure: Example: Bridge distributions with (left) and without (right) skewness tempering. Additionally deflating $\alpha_{t,i}$ reduces N_ϕ from 7 to 3. The optimal ϕ_0 is much closer to 1.

4. Estimation Results

- The model is estimated on the same data set as used by Adrian et al. (2019).
- Set $nfci_t$ as exogenous driving variable for gdp_{t+1} in measurement and state equations.
- Tuning: $N = 20000$ draws of the MH-Sampler with $M = 10000$ particles and $\Delta_r = 0.01$.

4.1 Static Parameters

Model Parameter	Mean	SD	q16	q84	q05	q95
γ_0	2.285	0.398	1.898	2.672	1.623	2.94
$\boldsymbol{\gamma_1}$	**-0.686**	0.362	-1.045	-0.335	**-1.311**	**-0.119**
$\delta_{1,0}$	0.865	0.285	0.573	1.164	0.446	1.372
$\boldsymbol{\delta_{1,1}}$	**0.242**	0.096	0.147	0.338	**0.102**	**0.412**
$\beta_{1,2}$	0.108	0.278	-0.192	0.396	-0.375	0.522
$\delta_{2,0}$	0.218	0.221	0.006	0.429	-0.143	0.595
$\boldsymbol{\delta_{2,1}}$	**-0.290**	0.226	**-0.477**	**-0.103**	-0.603	0.042
σ_{ν_1}	0.092	0.059	0.037	0.14	0.023	0.209
σ_{ν_2}	0.020	0.020	0.006	0.032	0.004	0.058

Table: Posterior Means, Standard Deviations (SD) and 68% and 90% Credible Sets

- Estimated parameter values indicate effect of $nfci_t$ on mean and variance that are significant based on 90% credible sets.
- Negative effect on the skewness holds only based on the 68% credible set.

4.2 Time-varying Parameters

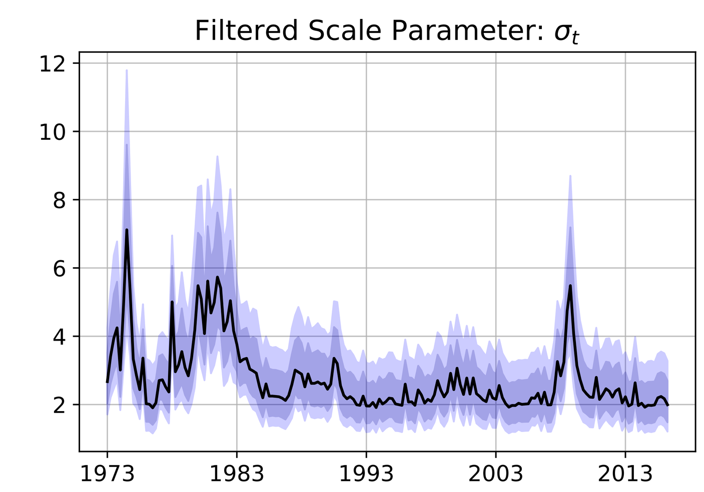

Figure: Volatility over time

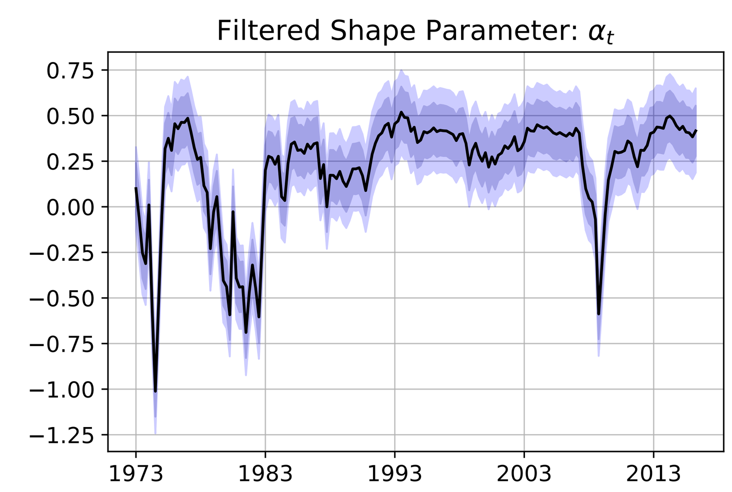

Figure: Skewness over time

- Volatility and downside risk increase in the 1980s and during the Great Recession.
- The estimated state of α_t also exhibits positive skewness in times of moderation similar to the findings of delle Monache et al. (2021).

4.3 Conditional Forecast Densities

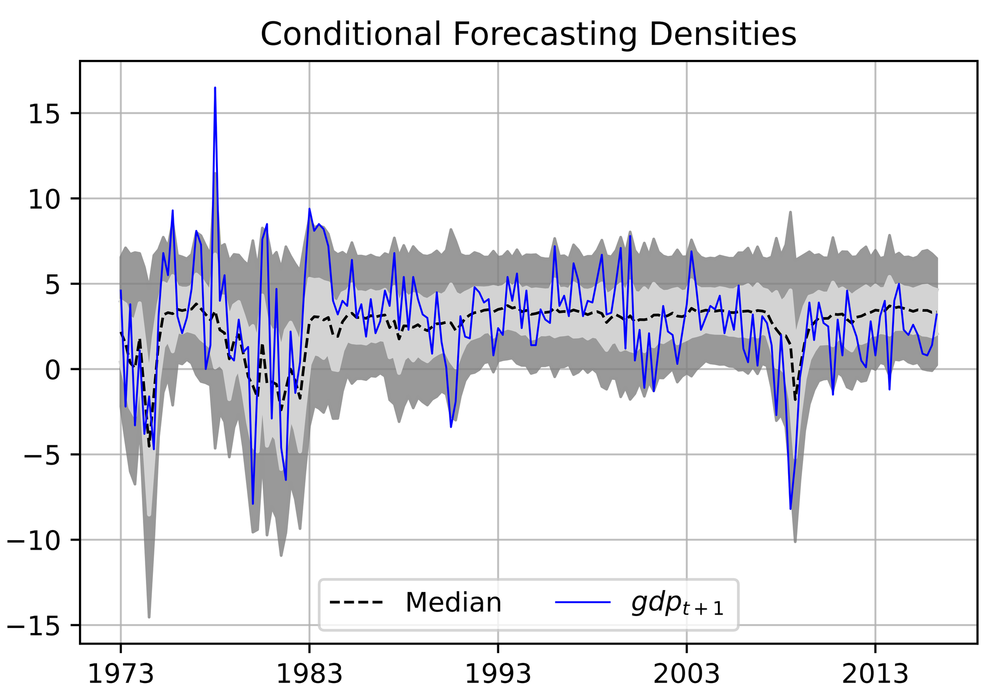

Figure: Estimated lower and upper 5% and 25% quantiles of the conditional forecast densities of gdp_{t+1} over time.

- While the upper quantiles remain relatively stable, the lower quantiles vary strongly over time indicating increased downside risk to GDP growth in times of financial distress.

5. Role of Skewness

To further investigate the importance of asymmetries in the conditional densities, I compare the SSV model with a symmetric Stochastic Volatility (SV) model:

- Bayes Factor and the log marginal data densities give decisive evidence for for the SSV model ($\mathcal{M}_1$) over the symmetric SV model ($\mathcal{M}_2$):

Bayes Factor	log Odds	$\log p(y\|\mathcal{M}_1)$	$\log p(y\|\mathcal{M}_2)$
1612.18	7.39	-435.78	-443.16

- Comparing the upside and downside entropy of the two models, reveals that downside entropy for the SSV model is much higher during economic crisis, indicating that modelling asymmetries matters especially to capture risks to the lower tails.

6. Conclusion

- I propose a Skewed Stochastic Volatility model to analyze Growth at Risk and conduct statistical inference on the estimated parameters.
- Bayesian model estimation using a tempered Particle Metropolis Hastings algorithm. The tempering schedule of the tempered particle filter is adapted to asymmetric distributions.
- Estimated parameters imply positive effect of $nfci_t$ on the variance and negative impact on mean and skewness of the conditional distributions of gdp_{t+1}.
- The SSV model is strongly favored over a symmetric SV model based on a Bayes Ratio of 1612.

References

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