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Jakub Growiec, Peter McAdam, Jakub Mućk On the optimal labor income share

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Abstract

Labor's share of income has attracted interest in recent years reflecting its apparent decline. These falls, witnessed across many countries, are usually deemed undesirable. Any such assertion, however, begs the question of what is the socially optimal labor share. We address this question using a micro-founded endogenous growth model calibrated on US data. We find that in our central calibration the socially optimal labor share is 17% (11 pp) above the decentralized equilibrium, calibrated to match the average observed in history. We also study the dependence of both long-run growth equilibria on model parameters and relate our results to Piketty's "laws of Capitalism". Finally, we demonstrate that cyclical movements in factor income shares are socially optimal and that the decentralized equilibrium typically does not generate excess volatility.

Keywords: Labor income share, Endogenous growth, Factor augmenting endogenous technical change, Social optimum, Decentralized allocation.

JEL Codes: O33, O41

Non technical summary

Labor's share of income has fascinated and thrilled both policymakers and researchers in recent years reflecting its apparent substantial decline. These falls, witnessed across many countries, are usually deemed undesirable. Any such assertion, however, begs the question of what is the socially optimal labor share. Despite its importance, the conclusions for this variable have not yet been drawn in the literature. Our objective is to bridge that knowledge gap.

We address this question using a micro-founded non-scale endogenous growth model. In our framework we combine two R&D sectors producing capital as well as labor augmenting innovations. The model economy uses the Dixit-Stiglitz monopolistic competition setup and the increasing variety framework of the R&D sector. Both the social planner (SP) and the decentralized allocation (DA) are solved for and compared.

We find that in our central calibration to the US economy the socially optimal labor share is 17% (11 pp) above the decentralized equilibrium. The same components which drive the wedge between the social and decentralized growth rate, reappear in the labor share analysis: markets fail to account for the external effects of physical capital on R&D activity; the shadow price of capital in the SP is replaced by its market price which accounts for imperfectly competitive mark-ups; markets fail to internalize the labor augmenting R&D duplication effects, and fail to account for the external effects of accumulating knowledge on future R&D. Given these frictions, we find that if the elasticity of factor substitution (σ) is below unity (as the bulk of empirical evidence suggests for the US and other developed economies), then the decentralized labor share is indeed socially suboptimal.

In addition, we document that the only economic parameter which changes the implication for the labor share is the degree of factor substitution. Our finding is therefore a very strong and robust result. Importantly, the case of gross substitutes tends to yield highly counter-factual outcomes otherwise, rendering it (in the context of this model) a somewhat unrealistic benchmark. For example an elasticity of $\sigma = 1.25$ (i.e., only marginally above Cobb Douglas), does produce a decentralized labor share above that of the social planner, but then the latter is as low as 0.52. Moreover such a mild perturbation away from CobbDouglas already produces equilibrium per capital growth rates of around 6%. Thus, the mantra that “growth is always good for labor” is conditional on which technological frictions one models, and what is the degree of factor substitution in the economy.

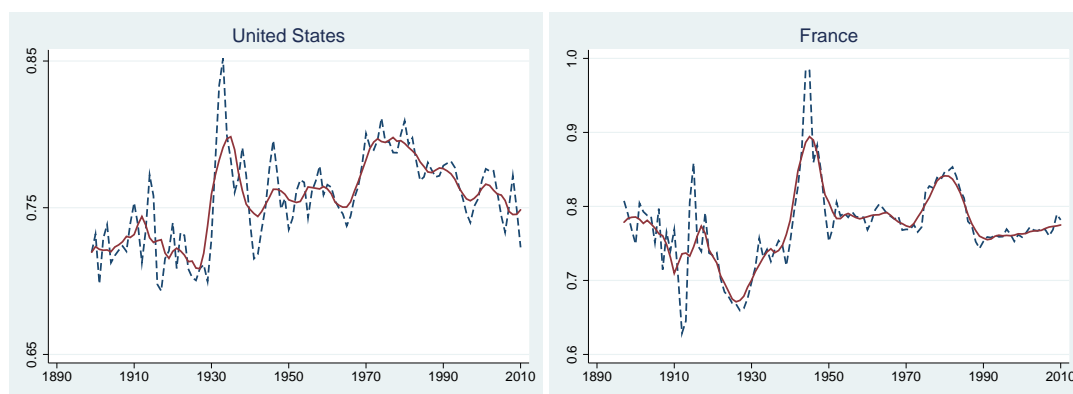
A final interesting point is that both the decentralized and social optimum are characterized by cycles. At the high frequency end this covers technology adoption costs, but in the longer run it reflects the tension between different R&D sectors and their effect on factor shares. There is some support for the statement that the decentralized equilibrium is likely to feature greater labor share volatility compared to the social optimum. We also find that implications of the model are at odds with Piketty's two “laws of Capitalism”.

1 Introduction

Although interest in labor's share of income has a long tradition in economics, current interest has crystalized around its apparent fall in recent decades. Such falls, witnessed across many countries, are often presented in normative terms: namely as undesirable, potentially requiring intervention. Any such assertion, however, begs the question of what is the appropriate point of comparison. Can we say, for instance, if there is a socially optimal labor share?

Moreover, what of its dynamics? We know from historical sources such as Piketty and Zucman (2014) that labor income shares – even over extremely long horizons (e.g., > 110 years) – can fluctuate considerably (see **Figure 1** for the US and France). Both aspects matter for any normative discussion on the labor share. For instance if the labor share is falling, yet still above its 'optimal' level (or fluctuating around it) then, arguably, this might be interpreted passively, as a manifestation of recognized fluctuations in factor shares (e.g., Mućk et al. (2018)). Moreover, in so far as long and persistent fluctuations in the labor share are observed in practice, we might also wonder whether such fluctuations are socially optimal. By comparison, can we say if the decentralized labor share is characterized by excessive volatility?¹

FIGURE 1: HISTORICAL LABOR SHARE: US (1899-2010) & FRANCE (1897-2010)



Notes: The French data is taken from Piketty and Zucman (2014). The US data also taken from Piketty and Zucman (2014) over the sample 1929-2010; prior to 1929 the labor share is extrapolated using the database by Groth and Madsen (2016) which provide compensation of employees and value added data starting in 1898 based on historical source provided by Liesner (1989). The **blue dashed line** is the level of the labor income share and the **red line** is a simple moving average process approximating its trend characteristics: $\frac{1}{10} \sum_{j=-5}^4 l_{s_{t-j}}$, where l_{s_t} is the labor share.

Remarkably enough, as far as we know, there is no investigation of these issues in the literature. This benign neglect contrasts with equivalent discussions in the growth literature: since Ramsey (1928) the question of whether society saves or produces 'too little' (relative to a social planner) is fundamental (e.g., de La Grandville, 2012). En-

¹In many economic literatures – e.g., in the real business cycle literature – it is often claimed that the presence of economic fluctuations (primarily at the business cycle frequency) is socially optimal: namely, efficient response to exogenous changes in the real economy, e.g., Cooley (1995).

ogenous growth theory², in addition, typically suggests that the socially optimal level of activity (in particular, R&D expenditures) exceeds the decentralized one, reflecting the absence of market distortions and externalities (e.g., Jones and Williams, 2000; Alvarez-Pelaez and Groth, 2005). But what of the labor share? Does ‘too little’ output in the decentralized economy imply too little labor share? *Ex ante*, it is by no means obvious. Given widespread interest in the labor share, this constitutes an important gap in our knowledge. Our paper addresses this gap.³

The structure of the paper is as follows. **Section 2** describes the model. We present a non-scale model of endogenous growth with two R&D sectors, giving rise to capital as well as labor augmenting innovations, drawing from the seminal contributions of Romer (1990) and Acemoglu (2003), and extending the model presented in Growiec et al. (2018). The model economy uses the Dixit-Stiglitz monopolistic competition setup and the increasing variety framework of the R&D sector. Both the social planner and decentralized allocations are solved for and compared. We see that the presence of markups arising from imperfect competition and externalities relating to R&D activity, are key points of comparison.

Section 3 calibrates the model to US data. We assume that a range of long-run averages from US data correspond to the decentralized Balanced Growth Path (BGP) of the model. Thereafter, in **Section 4** we solve and compare the BGP of each allocation. We list the channels and assumptions underlying the differences between both allocations. We also study the dependence of both long-run growth equilibria on model parameters and relate our results to Piketty’s “laws of Capitalism”. Following Growiec et al. (2018) we also consider the dynamic properties of the model around the balanced growth path (both in the decentralized and optimal allocation) in terms of oscillatory dynamics.

We find that – assuming production factors are gross complements (i.e., that the elasticity of factor substitutes is below unity) – the decentralized labor share is indeed socially suboptimal. The average difference, moreover, is large: about 17% (11 pp). This finding is interesting in itself given ongoing debates over labor equity, and their causes, (Atkinson, 2015). We describe the mechanisms which underly this wedge. For robustness, though, we also consider production characterized by Cobb Douglas and gross substitutes. In the latter case, and almost only in that case, the socially optimal labor share falls below the decentralized one. However already even for only a mild degree of gross substitutability, the labor share value is counter-factually low (at around 0.5) and also associated with excessively high per-capita growth rates. **Section 5** concludes.

²For textbook overviews, see Barro and Sala-i-Martin (2003); Acemoglu (2009).

³Note, we do not only study the implications for the labor share, but also the growth rate, employment in the research sector, consumption and capital accumulation, etc. Furthermore, by providing a quantitative comparison between the decentralized and the socially optimal factor shares in an endogenous R&D-based growth setup, this paper contributes also to the literature on distributional effects of innovation (Aghion et al., 2015; Jones and Kim, 2018).

2 Model

The framework is a generalization of Acemoglu (2003) with capital and labor augmenting R&D, building on the earlier induced innovation literature from Kennedy (1964) onwards as well as general innovation in monopolistic competition and growth literatures (Dixit and Stiglitz (1977), Romer (1990) and Jones (1999) and so on).

By ‘generalization’, we mean that we relax a number of features to make our conclusions more applicable to the question studied, as well as to correct for some counterfactual features (such as ‘scale’ effects). Formally, (i) our model is non-scale: both R&D functions are specified in terms of percentages of population employed in either R&D sector; (ii) we also assume R&D workers are drawn from the same pool as production workers⁴; (iii) we assume more general R&D technologies which allow for mutual spillovers between both R&D sectors (cf. Li, 2000) and for concavity in capital augmenting technical change; (iv) in contrast to Acemoglu (2003), the BGP growth rate in our model depends on preferences via employment in aggregate production. The tradeoff is due to drawing researchers from the same employment pool as production workers (a tradeoff not present in his model) and; (v) we use *normalized* CES production functions⁵ which, importantly, ensures valid comparative static comparisons in the elasticity of factor substitution.

First, though, we start with the simpler benchmark of the social planner model.⁶

2.1 The Social Planner’s Problem

The social planner maximizes the representative household’s utility from discounted consumption, given standard CRRRA preferences, (1), subject to the budget constraint (2) (i.e., the equation of motion of the aggregate per-capita capital stock), the ‘normalized’ production function, (3), the two R&D technologies (4)–(5) and the labor market clearing condition, (6):⁷

$$\max \int_0^{\infty} \frac{c^{1-\gamma} - 1}{1-\gamma} e^{-(\rho-n)t} dt \quad s.t. \quad (1)$$

⁴Acemoglu (2003) assumes that labor supply in the production sector is inelastic and R&D is carried out by a separate group of “scientists” who cannot engage in production labor. Our assumption affects the tension between both R&D sectors by providing R&D workers with a third option, the production sector.

⁵Normalization essentially implies representing the production relations in consistent index number form. Its parameters then have a direct economic interpretation. Otherwise the parameters can be shown to be scale dependent (i.e., a circular function of σ itself, as well as a function of the implicit normalization points). Subscripts 0 denote the specific normalization points: geometric (arithmetic) averages for non-stationary (stationary) variables. See de La Grandville (1989); Klump and de La Grandville (2000); Klump and Preissler (2000) for the seminal theoretical contributions. In our case, normalization is essentially important since comparative statics on production function parameters are a key concern.

⁶It is simpler because solving under the social optimum we can impose symmetry directly and deal in terms of aggregates, see Bénassy (1998).

⁷There are three control (c, ℓ_a, ℓ_b) and three state variables (k, λ_a, λ_b), in this optimization problem.

$$\dot{k} = y - c - (\delta + n)k - \zeta \dot{a}, \quad (2)$$

$$y = y_0 \left(\pi_0 \left(\lambda_b \frac{k}{k_0} \right)^\xi + (1 - \pi_0) \left(\frac{\lambda_a}{\lambda_{a0}} \frac{\ell_Y}{\ell_{Y0}} \right)^\xi \right)^{1/\xi} \quad (3)$$

$$\dot{\lambda}_a = A \left(\lambda_a \lambda_b^\phi x^{\eta_a} \ell_a^{\nu_a} \right), \quad (4)$$

$$\dot{\lambda}_b = B \left(\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b} \right) - d\lambda_b, \quad (5)$$

$$1 = \ell_a + \ell_b + \ell_Y, \quad (6)$$

where $\gamma > 0$ is the inverse of the intertemporal elasticity of substitution, $\rho > 0$ is the rate of time preference, and $n > 0$ is the (exogenous) growth rate of the labor supply.

Parameter ξ is given by $\xi = \frac{\sigma-1}{\sigma}$ where $\sigma \in [0, \infty)$ is the elasticity of factor substitution. CES function (3) nests the linear, Cobb–Douglas, and Leontief forms respectively when $\xi = 1, 0, -\infty$ (equivalently, $\sigma = \infty, 1, 0$). This parameter, important in many contexts⁸, turns out also to be critical in our analysis of the optimality and dynamics of the labor income share – relating, in particular, to whether factors are gross complements, i.e., $\sigma < 1$, or gross substitutes.

In (2) and (3), $y = Y/L$ and $k = K/L$ (i.e., output and capital per capita), where L is total employment and ℓ_a and ℓ_b are the shares (or “research intensity”) employed in labor and capital augmenting R&D (i.e., λ_a and λ_b) respectively, with $\ell_Y L = L_Y$, etc. We assume that capital augmentation is subject to gradual decay at rate $d > 0$, which mirrors susceptibility to obsolescence and embodied character of capital augmenting technologies, Solow (1960). This assumption is critical for the asymptotic constancy of unit capital productivity λ_b in the model, and thus for the existence of a BGP with purely labor augmenting technical change.

Factor augmenting innovations are created endogenously by the respective R&D sectors (Acemoglu, 2003), augmenting the technology menu by increasing the underlying parameters λ_a, λ_b , as in (4) and (5). Parameters A and B capture the unit productivity of the labor and capital augmenting R&D process, respectively; ϕ captures the spillover from capital to labor augmenting R&D;⁹ and ω measures the degree of decreasing returns to scale in capital augmenting R&D. By assuming $\omega \in (0, 1)$ we allow for the “standing on shoulders” effect in capital augmenting R&D, albeit we limit its scope insofar as it is less than proportional to the existing technology stock (Jones, 1995).

⁸The value of the elasticity of factor substitution has been shown to be a key parameter in many economic fields: e.g., the gains from trade (Saam, 2008); the strength of extensive growth (de La Grandville, 2016); multiple growth equilibria, development traps and indeterminacy (Azariadis, 1996; Klump, 2002; Kaas and von Thadden, 2003; Guo and Lansing, 2009), the responsive of investment and labor demand to various policy changes and shocks (Rowthorn, 1999; Chirinko, 2002) etc.

⁹We assume $\phi > 0$ indicating that more efficient use of physical capital also increases the productivity of labor augmenting R&D. Observe, there are *mutual* spillovers between both R&D sectors, with no prior restriction on their strength: $\dot{\lambda}_a = A \lambda_a^{1-\eta_a} \lambda_b^{\phi+\eta_a} k^{\eta_a} \ell_a^{\nu_a}$ and $\dot{\lambda}_b = B \lambda_a^{-\eta_b} \lambda_b^{1-\omega+\eta_b} k^{\eta_b} \ell_b^{\nu_b} - d\lambda_b$.

Finally, R&D activity may be subject to duplication externalities; the greater the number of researchers searching for new ideas, the more likely is duplication. Thus research effort may be characterized by diminishing returns, Kortum (1993). This is captured by parameters $\nu_a, \nu_b \in (0, 1]$: the higher is ν the lower the extent of duplication.¹⁰

Term $x \equiv k(\lambda_b/\lambda_a)$ captures the technology-corrected degree of capital augmentation of the workplace. This “lab equipment” term will to be constant along the BGP. The long-term endogenous growth engine is located in the linear labor augmenting R&D equation. To fulfill the requirement of the existence of a BGP along which the growth rates of λ_a and λ_b are constant, we assume that $\eta_b\phi + \eta_a\omega \neq 0$.¹²

The last term in (2) captures a negative externality that arises from implementing new labor augmenting technologies, with $\zeta \geq 0$. Motivated by León-Ledesma and Satchi (2018), we allow for a non-negative cost of adopting new labor augmenting technologies: since workers (as opposed to machines) need to develop skills compatible with each new technology, it is assumed that there is an external capital cost of such technology shifts (training costs, learning-by-doing, etc.). We posit that new capital investments are diminished by $\zeta\dot{a}$, where $\dot{a} = g\lambda_a \left(\frac{\pi}{\pi_0}\right)^{1/\alpha}$, g being the economic growth rate (Growiec et al., 2018).

2.2 Decentralized Allocation

The construction of the decentralized allocation draws from Romer (1990), Acemoglu (2003), Jones (2005), and Growiec et al. (2018). In particular, we use the Dixit and Stiglitz (1977) monopolistic competition setup and the increasing variety framework of the R&D sector. The general equilibrium is obtained as an outcome of the interplay between: households; final goods producers; aggregators of bundles of capital and labor intensive intermediate goods; monopolistically competitive producers of differentiated capital and labor intensive intermediate goods; and competitive capital and labor augmenting R&D firms.

2.2.1 Households

Analogous to the social planner’s allocation (SP), we assume that the representative household maximizes discounted CRRA utility:

$$\max \int_0^{\infty} \frac{c^{1-\gamma} - 1}{1-\gamma} e^{-(\rho-n)t} dt \quad (7)$$

¹⁰Observe that switching off all externalities and spillovers in (4)–(5) by setting $d = \omega = \eta_a = \eta_b = 0$ and $\nu_a = \nu_b = 1$ retrieves the original specification of R&D in Acemoglu (2003).¹¹ Moreover, compared with models which use Cobb Douglas production, equation (5) is akin to Jones’ (1995) formulation of the R&D sector, generalized by adding obsolescence and the lab equipment term. Thus, setting $d = \eta_b = 0$ retrieves Jones’ original specification. And (4) is the same as in Romer (1990) but scale-free (it features a term in ℓ_b instead of $\ell_b \cdot L$) and with lab equipment and a direct spillover from λ_b ; setting $\phi = \eta_a = 0$ retrieves the scale-free version of Romer (1990), cf. Jones (1999).

¹²All our qualitative results also go through for the special case $\eta_a = \eta_b = 0$, which fully excludes “lab equipment” terms in R&D. The current inequality condition is not required in such cases.

subject to the budget constraint:

$$\dot{v} = (r - \delta - n)v + w - c, \quad (8)$$

where $v = V/L$ is the household's per-capita holding of assets, $V = K + p_a \lambda_a + p_b \lambda_b$. The representative household is the owner of all capital and also holds the shares of monopolistic producers of differentiated capital and labor intensive intermediate goods. Capital is rented at a net market rental rate equal to the gross rental rate less depreciation: $r - \delta$. Solving the household's optimization problem yields the familiar Euler equation:

$$\hat{c} = \frac{1}{\gamma}(r - \delta - \rho), \quad (9)$$

where $\hat{c} = \dot{c}/c = g$ is the per-capita growth rate ("hats" denote growth rates.)

2.2.2 Final Goods Producers

The role of final goods producers is to generate the output of final goods (which are then either consumed by the representative household or saved and invested, leading to physical capital accumulation), taking bundles of capital and labor intensive intermediate goods as inputs. They operate in a perfectly competitive environment, where both bundles are remunerated at market rates p_K and p_L , respectively.

The final goods producers operate a normalized CES technology:

$$Y = Y_0 \left(\pi_0 \left(\frac{Y_K}{Y_{K0}} \right)^\xi + (1 - \pi_0) \left(\frac{Y_L}{Y_{L0}} \right)^\xi \right)^{\frac{1}{\xi}}. \quad (10)$$

The optimality condition implies that final goods producers' demand for capital and labor intensive intermediate goods bundles satisfies:

$$\frac{p_K}{p_L} = \frac{\pi}{1 - \pi} \frac{Y_L}{Y_K}, \quad (11)$$

where $\pi = \pi_0 \left(\frac{Y_K}{Y_{K0}} \frac{Y_0}{Y} \right)^\xi$ is the elasticity of final output with respect to Y_K .

2.2.3 Aggregators of Capital and Labor Intensive Intermediate Goods

There are two symmetric sectors whose role is to aggregate the differentiated (capital or labor intensive) goods into the bundles Y_K and Y_L demanded by final goods producers. It is assumed that the differentiated goods are imperfectly substitutable (albeit gross substitutes). The degree of substitutability is captured by parameter $\varepsilon \in (0, 1)$:

$$Y_K = \left(\int_0^{N_K} X_{K_i}^\varepsilon di \right)^{\frac{1}{\varepsilon}}. \quad (12)$$

Aggregators operate in a perfectly competitive environment and decide upon their demand for intermediate goods, the price of which will be set by the respective mo-

nopolistic producers (discussed in the following subsection).

For capital intensive bundles, the aggregators maximize:

$$\max_{X_{Ki}} \left\{ p_K \left(\int_0^{N_K} X_{Ki}^\varepsilon di \right)^{\frac{1}{\varepsilon}} - \int_0^{N_K} p_{Ki} X_{Ki} di \right\}. \quad (13)$$

There is a continuum of measure N_K of capital intensive intermediate goods producers. Optimization implies the following demand curve:

$$X_{Ki} = x_K(p_{Ki}) = \left(\frac{p_{Ki}}{p_K} \right)^{\frac{1}{\varepsilon-1}} Y_K^{\frac{1}{\varepsilon}}. \quad (14)$$

Equivalent terms follow for labor intensive intermediate goods producers.

2.2.4 Producers of Differentiated Intermediate Goods

It is assumed that each of the differentiated capital or labor intensive intermediate goods producers, indexed by $i \in [0, N_K]$ or $i \in [0, N_L]$ respectively, has monopoly over its specific variety. It is therefore free to choose its preferred price p_{Ki} or p_{Li} . These firms operate a simple linear technology, employing either only capital or only labor.

For the case of capital intensive intermediate goods producers, the production function is $X_{Ki} = K_i$. Capital is rented at the gross rental rate r . The optimization problem is:

$$\max_{p_{Ki}} (p_{Ki} X_{Ki} - r K_i) = \max_{p_{Ki}} (p_{Ki} - r) x_K(p_{Ki}). \quad (15)$$

The optimal solution implies $p_{Ki} = r/\varepsilon$ for all $i \in [0, N_K]$. This implies symmetry across all differentiated goods: they are sold at equal prices, thus their supply is also identical, $X_{Ki} = \bar{X}_K$ for all i . Market clearing implies:

$$K = \int_0^{N_K} K_i di = \int_0^{N_K} X_{Ki} di = N_K \bar{X}_K \quad Y_K = N_K^{\frac{1-\varepsilon}{\varepsilon}} K. \quad (16)$$

The demand curve implies that the price of intermediate goods is linked to the price of the capital intensive bundle as in $p_K = p_{Ki} N_K^{\frac{\varepsilon-1}{\varepsilon}} = \frac{r}{\varepsilon} N_K^{\frac{\varepsilon-1}{\varepsilon}}$.

The *labor intensive* sector follows symmetrically: $X_{Li} = L_{Yi}$, $L_Y = \ell_Y L = \int_0^{N_L} L_{Yi} di$, and $p_{Li} = w/\varepsilon$, $p_L = p_{Li} N_L^{\frac{\varepsilon-1}{\varepsilon}} = \frac{w}{\varepsilon} N_L^{\frac{\varepsilon-1}{\varepsilon}}$, where w is the market wage rate.

Aggregating across all intermediate goods producers, we obtain that their total profits are equal to $\Pi_K N_K = rK \left(\frac{1-\varepsilon}{\varepsilon} \right)$ and $\Pi_L N_L = wL_Y \left(\frac{1-\varepsilon}{\varepsilon} \right)$ for capital and labor intensive goods respectively. Streams of profits per person in the representative household are thus $\pi_K = \Pi_K/L$ and $\pi_L = \Pi_L/L$, respectively. Hence, the total remuneration channeled to the capital intensive sector equals $p_K Y_K = \frac{r}{\varepsilon} K = rK + \Pi_K N_K$, whereas the total remuneration channeled to the labor intensive sector equals $p_L Y_L =$

$\frac{w}{\varepsilon}L_Y = rL_Y + \Pi_L N_L$. In equilibrium, factor shares then amount to,

$$\pi = \pi_0 \left(\frac{KY_0}{YK_0} \right)^\xi \left(\frac{N_K}{N_{K0}} \right)^{\xi \left(\frac{1-\varepsilon}{\varepsilon} \right)}, \quad (17)$$

$$1 - \pi = (1 - \pi_0) \left(\frac{Y_0 L_Y}{Y L_{Y0}} \right)^\xi \left(\frac{N_L}{N_{L0}} \right)^{\xi \left(\frac{1-\varepsilon}{\varepsilon} \right)}. \quad (18)$$

Incorporating all these choices into (10), and using the definitions $\lambda_b = N_K^{\frac{1-\varepsilon}{\varepsilon}}$ and $\lambda_a = N_L^{\frac{1-\varepsilon}{\varepsilon}}$ retrieves production function (3).

2.2.5 Capital and Labor Augmenting R&D Firms

The role of capital and labor augmenting R&D firms is to produce innovations which increase the variety of available differentiated intermediate goods, either N_K or N_L , and thus indirectly also λ_b and λ_a . Patents never expire, and patent protection is perfect. R&D firms sell these patents to the representative household which sets up a monopoly for each new variety. Patent price, p_b or p_a , which reflects the discounted stream of future monopoly profits, is set at the competitive market. There is free entry to R&D.

R&D firms employ labor only: $L_a = \ell_a L$ and $L_b = \ell_b L$ workers are employed in the labor and capital augmenting R&D sectors, respectively. There is also an externality from the total physical capital stock working through the “lab equipment” term in the R&D production function. Furthermore, the R&D firms perceive their production technology as linear in labor, while in fact it is concave due to duplication externalities.

Incorporating these assumptions and using the notion $x \equiv \lambda_b k / \lambda_a$, capital augmenting R&D firms maximize:

$$\max_{\ell_b} (p_b \dot{\lambda}_b - w \ell_b) = \max_{\ell_b} ((p_b Q_K - w) \ell_b), \quad (19)$$

where $Q_K = B (\lambda_b^{1-\omega} x^{\eta_b} \ell_b^{\nu_b-1})$ is treated by firms as a constant in the steady state (Romer, 1990; Jones, 2005). Analogously, labor augmenting R&D firms maximize:

$$\max_{\ell_a} (p_a \dot{\lambda}_a - w \ell_a) = \max_{\ell_a} ((p_a Q_L - w) \ell_a), \quad (20)$$

where $Q_L = A (\lambda_a \lambda_b^\phi x^{\eta_a} \ell_a^{\nu_a-1})$ is treated as exogenous.

Free entry into both R&D sectors implies $w = p_b Q_K = p_a Q_L$. Purchase of a patent entitles the holders to a per-capita stream of profits equal to π_K and π_L , respectively. While the production of any labor augmenting varieties lasts forever, there is a constant rate d at which production of capital intensive varieties becomes *obsolete*. This effect is external to patent holders and thus is not strategically taken into account when accumulating the patent stock.¹³

¹³In other words, by solving a static optimization problem, capital augmenting R&D firms do not take

2.2.6 Equilibrium

We define the *decentralized equilibrium* as the collection of time paths of all the respective quantities: $c, \ell_a, \ell_b, k, \lambda_b, \lambda_a, Y_K, Y_L, \{X_{Ki}\}, \{X_{Li}\}$ and prices $r, w, p_K, p_L, \{p_{Ki}\}, \{p_{Li}\}, p_a, p_b$ such that: (1) households maximize discounted utility subject to their budget constraint; (2) profit maximization is followed by final-goods producers, aggregators and producers of capital and labor intensive intermediate goods, and capital and labor augmenting R&D firms; (3) the labor market clears: $L_a + L_b + L_Y = (\ell_a + \ell_b + \ell_Y)L = L$; (4) the asset market clears: $V = vL = K + p_a\lambda_a + p_b\lambda_b$, where assets have equal returns: $r - \delta = \frac{\pi_L}{p_a} + \frac{\dot{p}_a}{p_a} = \frac{\pi_K}{p_b} + \frac{\dot{p}_b}{p_b} - d$; and, finally (5), such that the aggregate capital stock satisfies $\dot{K} = Y - C - \delta K - \zeta \dot{a}L$, where the last term is an externality term (as previously discussed).

2.3 Solving for the Social Planner Allocation

In this section, we first solve analytically for the BGP of the social planner allocation of our endogenous growth model and then linearize the implied dynamical system around the BGP.

2.3.1 Balanced Growth Path

Any neoclassical growth model can exhibit balanced growth only if technical change is purely labor augmenting or if production is Cobb Douglas, Uzawa (1961). This conclusion holds here too. Hence, once we presume a CES production function, the analysis of dynamic consequences of technical change, which is not purely labor augmenting, must be done outside the BGP.

Along the BGP, we obtain the following growth rate of key model variables:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a}, \quad (21)$$

where stars denote steady-state values. Hence, ultimately long-run growth is driven by labor augmenting R&D. This can be explained by the fact that labor is the only non-accumulable factor in the model, it is complementary to capital along the aggregate production function, and the labor augmenting R&D equation is linear with respect to λ_a . The following variables are constant along the BGP: $y/k, c/k, \ell_a, \ell_b$ and λ_b (i.e., asymptotically there is no capital augmenting technical change).

the dynamic (external) obsolescence effect into account.

2.3.2 Euler Equations

Having set up the Hamiltonian and computed its derivatives, the following Euler equations are obtained for the SP:¹⁴

$$\hat{c} = \frac{1}{\gamma} \left(\frac{y}{k} \left(\pi + \frac{1 - \pi}{\ell_Y} \left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b} \right) \right) - \delta - \rho \right), \quad (22)$$

$$\varphi_1 \hat{\ell}_a + \varphi_2 \hat{\ell}_b = Q_1, \quad (23)$$

$$\varphi_3 \hat{\ell}_a + \varphi_4 \hat{\ell}_b = Q_2, \quad (24)$$

where

$$\varphi_1 = \nu_a - 1 - (1 - \xi) \pi \frac{\ell_a}{\ell_Y}, \quad (25)$$

$$\varphi_2 = -(1 - \xi) \pi \frac{\ell_b}{\ell_Y} \quad (26)$$

$$\varphi_3 = -(1 - \xi) \pi \frac{\ell_a}{\ell_Y} \quad (27)$$

$$\varphi_4 = \nu_b - 1 - (1 - \xi) \pi \frac{\ell_b}{\ell_Y}, \quad (28)$$

$$Q_1 = -\gamma \hat{c} - \rho + n + \hat{\lambda}_a \left(\frac{\ell_Y \nu_a}{\ell_a} + 1 - \eta_a - \eta_b \frac{\ell_b \nu_a}{\ell_a \nu_b} \right) - \phi \hat{\lambda}_b + ((1 - \xi) \pi - \eta_a) \hat{x} \quad (29)$$

$$Q_2 = -\gamma \hat{c} - \rho + n + \hat{\lambda}_a + \hat{\lambda}_b \left(\frac{\pi}{1 - \pi} \frac{\ell_Y \nu_b}{\ell_b} + (\phi + \eta_a) \frac{\nu_b \ell_a}{\nu_a \ell_b} + \eta_b \right) + ((1 - \xi) \pi - \eta_b) \hat{x} + d \left(\frac{\pi}{1 - \pi} \frac{\ell_Y \nu_b}{\ell_b} + (\phi + \eta_a) \frac{\nu_b \ell_a}{\nu_a \ell_b} - \omega + \eta_b \right). \quad (30)$$

2.3.3 Steady State and Linearization of the Transformed System

The above Euler equations and dynamics of state variables are then rewritten in terms of *stationary* variables which are constant along the BGP, i.e., in coordinates: $u = (c/k), \ell_a, \ell_b, x, \lambda_b$, and with auxiliary variables $z = (y/k), \pi, g$. The full steady state of the transformed system is listed in [Appendix A.1](#). This non-linear system of equations is solved numerically, yielding a unique steady state of the de-trended system, and thus a unique BGP of the model in original variables. All further analysis of the social planner allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22)–(24), (2) and (5), taking the BGP equality (21) as given.

2.4 Solving for the Decentralized Allocation

When solving for the decentralized allocation, we broadly follow the steps carried out in the case of the social planner allocation. We first solve analytically for the BGP of our endogenous growth model and then linearize the implied dynamical system around the BGP.

¹⁴A sufficient condition for all transversality conditions to be satisfied in the social optimum (as well as in the decentralized equilibrium) is that $(1 - \gamma)g + n < \rho$.

2.4.1 Balanced Growth Path

Along the BGP, we obtain the following growth rate of the key model variables:

$$g = \hat{k} = \hat{c} = \hat{y} = \hat{w} = \hat{p}_b = \hat{p}_{Li} = \hat{\lambda}_a = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a}. \quad (31)$$

The following quantities are constant along the BGP: $y/k, c/k, \ell_a, \ell_b, Y_K/Y, Y_L/Y$ and λ_b (again, note, asymptotically, the absence of capital augmenting technical change). The following prices are also constant along the BGP: $r, p_a, p_K, p_L, \{p_{Ki}\}$.

2.4.2 Euler Equations

The decentralized equilibrium is associated with the following Euler equations describing the first-order conditions:

$$\hat{c} = \frac{1}{\gamma} \left(\varepsilon \pi \frac{y}{k} - \delta - \rho \right), \quad (22')$$

$$\varphi_1 \hat{\ell}_a + \varphi_2 \hat{\ell}_b = \tilde{Q}_1, \quad (23')$$

$$\varphi_3 \hat{\ell}_a + \varphi_4 \hat{\ell}_b = \tilde{Q}_2, \quad (24')$$

where

$$\tilde{Q}_1 = -\varepsilon \pi \frac{y}{k} + \delta + \hat{\lambda}_a \frac{\ell_Y}{\ell_a} - \phi \hat{\lambda}_b + ((1 - \xi)\pi - \eta_a) \hat{x} \quad (29')$$

$$\tilde{Q}_2 = -\varepsilon \pi \frac{y}{k} + \delta + \hat{\lambda}_a + (\hat{\lambda}_b + d) \left(\frac{\pi}{1 - \pi} \frac{\ell_Y}{\ell_b} \right) - \hat{\lambda}_b (1 - \omega) - d + ((1 - \xi)\pi - \eta_b) \hat{x} \quad (30')$$

and φ_1 through φ_4 are defined as in (25)–(28). The full steady state of the transformed system is listed in [Appendix A.2](#). All further analysis of the decentralized allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22')–(24'), (2) and (5), taking the BGP equality (31) as given.

2.4.3 Departures from the Social Optimum

Departures of the decentralized allocation from the optimal one can be tracked back to specific assumptions regarding the information structure of the decentralized allocation. Those differences are the following:

1. In the consumption Euler equation, comparing equations (22) with (22'), the term $\frac{y}{k} \left(\pi + \frac{1 - \pi}{\ell_Y} \left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b} \right) \right)$ is replaced by $\varepsilon \pi \frac{y}{k}$. This is due to two effects:
 - (a) in contrast to the social planner, markets fail to account for the external effects of physical capital on R&D activity via the lab equipment terms (with respective elasticities η_b and η_a);
 - (b) ε appears in the decentralized allocation due to imperfect competition in the labor and capital augmenting intermediate goods sectors.

2. In the Euler equations for ℓ_a and ℓ_b (equations (23), (24), (23'), (24')) the shadow price of physical capital $\hat{c} - \rho + n$ is replaced by its market price $r - \delta = \varepsilon\pi\frac{y}{k} - \delta$ which accounts for markups arising from imperfect competition.
3. In the Euler equation for ℓ_a , the term $\left(\frac{\ell_Y\nu_a}{\ell_a} + 1 - \eta_a - \eta_b\frac{\ell_b\nu_a}{\ell_a\nu_b}\right)$ is replaced by $\frac{\ell_Y}{\ell_a}$. This is due to two effects:
 - (a) ν_a is missing because markets fail to internalize the labor augmenting R&D duplication effects when $\nu_a < 1$;
 - b) the latter two components are missing because markets fail to account for the external effects of accumulating knowledge on future R&D productivity. These effects are included in the shadow prices of λ_a and λ_b in the social planner allocation but not in their respective market prices.
4. Analogously, in the Euler equation for ℓ_b , the term given by, $\frac{\ell_Y\nu_b}{\ell_b}\frac{\pi}{1-\pi} + (1 - \omega) + \eta_b + (\phi + \eta_a)\frac{\ell_a\nu_b}{\ell_b\nu_a}$ is replaced by $\frac{\ell_Y}{\ell_b}\frac{\pi}{1-\pi}$. The same reasoning follows as per point 3.

3 Calibration of the Model

The calibration for the decentralized model is listed in **Table 1**. We assume that a range of long-run averages from US data correspond to the decentralized BGP of the model (1929-2015). Doing so allows us to calibrate the rates of economic and population growth, capital productivity and income share, and the consumption-to-capital ratio. Likewise, we assign CES normalization parameters to match US long-run averages for factor income shares (we adjust the payroll share by proprietors' income, Mućk et al. (2018)). This implies an average labor share of 0.67.

Next, we turn to the elasticity of substitution between labor and capital (σ) which is the fundamental economic parameter in our analysis. We calibrate factors to be gross complements, i.e., $\xi < 0 \Leftrightarrow \sigma < 1$. This choice stems from the fact that the bulk of the empirical studies for the US aggregate production function document that σ is systematically below unity (Chirinko, 2008; Klump et al., 2012).¹⁵ Most of the empirical evidence exploiting time series variation for other countries also implies $\sigma < 1$ (McAdam and Willman, 2013; Mućk, 2017).

¹⁵For instance, the seminal Arrow et al. (1961) paper found an aggregate elasticity over 1909-1949 of 0.57 which, at the aggregate level, is confirmed by Antràs (2004). More recently, Klump et al. (2007) reported $\hat{\sigma} \approx 0.7$. The gross complementarity between factors is also confirmed at the industry (Herrendorf et al., 2015; Chirinko and Mallick, 2017) and firm level (Oberfield and Raval, 2018). Importantly, the elasticity uncovered is found systematically below unity even if more flexible functional forms of aggregate production function are considered (Growiec and Mućk, 2016).

TABLE 1: BASELINE CALIBRATION: PRE-DETERMINED PARAMETERS (DECENTRALIZED ALLOCATION)

Parameter	Symbol	Value	Source/Target
<i>Income and Production</i>			
GDP Per-Capita Growth	g	0.0171	geometric average of gross growth rates
Population Growth rate	n	0.0153	geometric average of gross growth rates
Labor in Aggregate Production	ℓ_Y, ℓ_Y^*	0.5394	$\frac{0.5(\gamma + \frac{\rho}{\sigma})}{1 + 0.5(\gamma + \frac{\rho}{\sigma})}$
Capital Productivity	z_0, z^*	0.3442	geometric average
Consumption-to-Capital	u^*	0.2199	geometric average
Capital Income Share	π_0, π^*	0.3260	arithmetic average
Depreciation	δ	0.0600	Caselli (2005)
Factor Substitution Parameter	ξ	-0.4286	$\Rightarrow \sigma = 0.7$, Klump et al. (2007)
Net Real Rate of Return	$r^* - \delta$	0.0499	$r^* - \delta = \gamma g + \rho$
Substitutability Between Intermediate Goods	ε	0.9793	$\varepsilon = \frac{r^*}{\pi^* z^*}$
<i>R&D Sectors</i>			
R&D Duplication Parameters	$\nu_a = \nu_b$	0.7500	Jones and Williams (2000)
Technology Augmenting Terms	$\lambda_{a0}, \lambda_{b0}$	1.0000	see text
Technology Augmenting Terms	λ_b^*	1.0000	$\lambda_b^* = \lambda_{b0} \frac{z^*}{z_0} \left(\frac{\pi^*}{\pi_0}\right)^{\frac{1}{\xi}}$
Employment Share in R&D sectors	ℓ_Y^a, ℓ_Y^b	0.2033	$\ell_a^* = \ell_b^*$ for $\ell_a^* + \ell_b^* = 1 - \ell_Y^*$
Lab-Equipment Term [†]	x_0, x^*	61.7900	$x^* = x_0 \frac{\ell_Y^*}{\ell_{Y,0}} \left(\frac{1}{1 - \pi_0} \left(\frac{z^* \lambda_{b0}}{z_0 \lambda_b^*}\right)^\xi - \frac{\pi_0}{1 - \pi_0}\right)^{-1/\xi}$
<i>Preferences</i>			
Inverse Intertemporal Elasticity of Substitution	γ	1.7500	Barro and Sala-i-Martin (2003)
Time Preference	ρ	0.0200	Barro and Sala-i-Martin (2003)

Notes: [†] $x_0 = \frac{\lambda_{b0} k_0}{\lambda_{a0}} = 61.79$.

However, the literature based predominantly on cross-country variation is rather inconclusive about the magnitude of σ . On one hand, several papers (Piketty, 2014; Piketty and Zucman, 2014; Karabarbounis and Neiman, 2014) employ gross substitutes. On the other, recent studies exploiting macro panels and allowing for factor augmentation in the supply-side system approach tend to return to the conclusion of gross complementarity (Villacorta, 2018; Mućk, 2017). Given this, we also consider $\sigma < 1$ as our benchmark, but $\sigma > 1$ is used in our robustness exercises.

Next, given this, four identities included in the system (see appendix equations A.9–A.17) drive the calibration of other parameters in a model-consistent manner: ℓ_Y^* , r^* , λ_b^* , x^* and ε . Employment in final production is also set in a model-consistent manner. In the absence of any other information, we agnostically assume that the share $1 - \ell_Y^*$ is split equally between employment in both R&D sectors. For the model-consistent value of ℓ_Y^* , this formula leads to values close to those typically considered for the non-routine cognitive occupational group (e.g., Jaimovich and Siu (2012), using BLS data, show this ratio to be between 29% and 38% (over 1982-2012)).

Regarding the preference parameters, for the (inverse) intertemporal elasticity of substitution and the rate of time preference, we use values typically found in the literature. For the duplication externalities, we agnostically assume $\nu_a = \nu_b = 0.75$ following the (albeit single R&D sector) value in Jones and Williams (2000). The technology augmenting term λ_b^* is set in a model-consistent manner, normalized to unity.

The final step is to assign values to the remaining parameters, in particular the technological parameters of the R&D equations. We do this by solving the four remaining equations in system (A.9)–(A.17) with respect to the remaining parameters, see **Table B.1**. Given this benchmark calibration, the steady state is a saddle point.

4 Is the Decentralized Labor Share Socially Optimal?

Given the model set up and its benchmark calibration, we can now come to our central question: is the decentralized labor share socially optimal? In **Table 2**, columns (1) and (2) show the decentralized allocation (DA) and social planner (SP) outcomes for our benchmark calibration; (3) and (4), considered later, alternatively impose Cobb–Douglas and gross substitutes.

Comparing column (1) with (2), we see that the BGP of the decentralized solution features lower growth, lower R&D activity, but higher consumption (u is higher). Moreover, with lower growth, capital costs are cheaper and the net real rate of return of capital is higher, and capital productivity is accordingly higher. That there should be a growth differential in favor of the social planner is straightforward; it follows from our discussion in **Section 2.4.3**: due to externalities, there is insufficient R&D activity and thus lower growth. But the comparison in terms of the labor share is perhaps less obvious. In fact we see the striking result that the labor share in the social optimum is around 17% (11 pp) *above* the decentralized allocation.

To understand why, let us decompose the capital income share, π , in the following

TABLE 2: BGP COMPARISON UNDER THE BASELINE CALIBRATION

Variable	Symbol	(1)	(2)	(3)	(4)
		DA	SP		
			Baseline	C-D	Piketty
		$\xi = -0.43$	$\xi = 0$	$\xi = 0.20$	
Output growth rate	g	0.0171	0.0339	0.0425	0.0581
Consumption-to-capital ratio	u^*	0.2199	0.1628	0.1180	0.0856
Capital productivity	z^*	0.3442	0.3071	0.2832	0.2743
Employment in production	ℓ_Y^*	0.5934	0.4385	0.4160	0.3854
Employment in labor augmenting R&D	ℓ_a^*	0.2033	0.2575	0.2447	0.2240
Employment in capital augmenting R&D	ℓ_b^*	0.2033	0.3040	0.3393	0.3906
Relative Share	ℓ_a^*/ℓ_b^*	1	0.8470	0.7212	0.5735
Labor income share	$1 - \pi^*$	0.6739	0.7854	0.6739	0.5243
Relative to DA (%)	$\frac{1-\pi^*DA}{1-\pi^*} - 1$	0	0.1655	0	-0.2220
Capital income share	π^*	0.3261	0.2146	0.3261	0.4757
Net real rate of return	$r^* - \delta$	0.0499	0.0059	0.0323	0.0704
Capital augmenting technology	λ_b^*	1.0000	2.3696	3.3162	5.2600
Lab equipment	x^*	61.7900	173.3363	342.7082	928.9625

two ways (recalling that $1 - \pi$ is the labor income share):

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b k}{k_0}\right)^\xi \left(\frac{y}{y_0}\right)^{-\xi} \Rightarrow \hat{\pi} = \xi(\hat{\lambda}_b + \hat{k} - \hat{y}), \quad (32)$$

$$\frac{\pi}{1 - \pi} = \frac{\pi_0}{1 - \pi_0} \left(\frac{x \ell_{Y0}}{x_0 \ell_Y}\right)^\xi \Rightarrow \hat{\pi} = \xi(1 - \pi)(\hat{x} - \hat{\ell}_Y). \quad (33)$$

Equation (32) shows that under gross complementarity ($\sigma < 1 \Leftrightarrow \xi < 0$), the capital share decreases with inverse capital productivity and with capital augmentation (i.e., the capital augmenting technology improvements are “labor biased”).

Equation (33), in turn, follows from the definition of the aggregate production function and the “lab equipment” term x . Given $\hat{\ell}_Y \equiv -\left(\frac{\ell_a}{\ell_Y} \hat{\ell}_a + \frac{\ell_b}{\ell_Y} \hat{\ell}_b\right)$, the dynamics of employment in the goods sector are equal to the inverse of the dynamics of total R&D employment. It then follows that dynamics of the labor share are *uniquely determined* by the sum of the dynamics of the lab equipment component and R&D employment. As before, the sign of this relationship depends upon the substitution elasticity: if $\xi < 0$ then increases in R&D intensity reduce π , and thus increase the labor share, and vice versa.

Comparing the decentralized and the social planner’s allocation through the lens of (32), we observe that the large difference in factor shares at the BGP is driven almost exclusively by the difference in the level of capital augmentation λ_b^* . This result suggests that technical change is quantitatively more important for explaining labor

share developments than the share of the capital stock in output.

Equivalently, by (33), this large difference in the degree of capital augmentation shows up in the lab equipment term x^* . It is also strengthened by the discrepancy in employment in final production ℓ_Y^* , which is higher in the decentralized allocation because the planner devotes more resources to (both types of) R&D. Thanks to this, coupled with relatively more saving, it achieves faster growth at the BGP but with a lower consumption-to-capital ratio and a lower rate of return to capital. All of these make for a higher labor share in the optimal allocation.

4.1 Impact of Parameter Variation on the Equilibrium Labor Share

The results just discussed hold for the benchmark calibration (of Table 1). Accordingly, we now consider sensitivity to deviations from that calibration. Figure 2 presents the impact of varying key selected model parameters, holding others constant, on the BGP level of the labor share. Again, we perform this exercise for both the decentralized and social optimum.

Essentially all panels can be interpreted through the lens of equations (32) and (33). As agents become less patient (higher ρ), R&D intensity falls, as does the labor share. Similar reasoning pertains to the inverse intertemporal elasticity of substitution γ . That $\frac{\partial(1-\pi)}{\partial\eta_b} > 0$ arises from the usual property that, under our gross-complements benchmark, improvements in capital augmenting technical change are labor biased; analogously $\frac{\partial(1-\pi)}{\partial\eta_a} < 0$. Likewise, we have under gross complements: $\frac{\partial(1-\pi)}{\partial\nu_a} > 0$, $\frac{\partial(1-\pi)}{\partial\nu_b} < 0$. If capital depreciates faster, the capital (labor) share rises (falls).

Finally, we see that under gross substitutes, $\xi > (\sigma > 1)$, the DA labor share *exceeds* that of the SP (see top left panel of figure 2).¹⁶ We discuss this case further below, but it is straightforward to motivate, since the previously-discussed mechanisms go into reverse; technical improvements tend to be capital biased.¹⁷ A more extensive study of the dependence of both BGPs on key model parameters ($\rho, \gamma, \nu_b, \eta_b$) is included in Figure C.1–Figure C.3. And the equivalent figure for the gross substitutes case is given in Figure C.4. They are essentially a mirror image of our benchmark gross complements case.

4.2 Impact of Parameter Variation on the Equilibrium Growth Rate

So far we have confirmed the received wisdom that g^{DA} is socially suboptimal. This appears to be generally true in our model. Focusing on a reasonable parameter support, and assuming gross complements (see Figure 3), we can however identify a few credible cases where the difference between the two growth rates becomes small:

¹⁶The second, albeit weaker case is when η_b , the exponent on the lab equipment term in capital augmenting technical progress, becomes very small.

¹⁷Note that the lack of dependence of the BGP on ξ in the decentralized allocation follows from CES normalization (Klump and de La Grandville, 2000), coupled with the fact that we have calibrated the normalization constants to the BGP of the decentralized allocation.

1. A higher ρ (i.e., more impatience for current consumption), implies less capital and R&D accumulation and lower equilibrium growth than otherwise. If $\rho^{SP} \gg \rho^{DA}$ then $g^{SP} \rightarrow_+ g^{DA}$.¹⁸
2. If the consumption smoothing motive is sufficiently weak (γ high) then $g^{SP} \rightarrow_+ g^{DA}$.
3. If the lab equipment exponents are weak, $\nu_a \rightarrow 0$ or $\nu_b \rightarrow 0$, then they attenuate the engine of long run growth in the R&D equations and thus pull *both* g^{SP} and g^{DA} down.

Finally, note, departing from gross substitutes, we see the dramatic result that as σ (or equivalently ξ) increases, the gap $g^{SP} - g^{DA}$ exponentially widens; conversely it narrows as substitution possibilities tend to zero (the Leontief case). We explore this in the next section.

4.3 Impact of Elasticity of Substitution Variation on the BGP

Although we regard the gross complements case to be the more empirically relevant (at least for the aggregate US economy), we also investigate the Cobb Douglas and gross substitutes case. Accordingly, the SP is solved anew and presented under columns (3) and (4) of **Table 2**, respectively. The effect of a continuous variation in the substitution elasticity itself is moreover graphed in **Figure 4**.

Both alternative parametrizations are markedly more growth friendly. Per-capita output grows at the counterfactual rate of around 4 – 6% (specifically, 4.25 – 5.80%), exceeding both the previous SP and DA by a large margin, with an inflection point at $\xi \approx 0.25$, after which per-capita growth shoots through the roof. That steady state per-capita growth is an increasing function of the substitution elasticity, though, is to be expected. Intuitively, easier factor substitution – by staving off diminishing returns – can prolong extensive growth (i.e., scarce factors can be substituted by abundant ones). The formal proof of this¹⁹ can be though related through the properties of the normalized CES function as a General Mean function.²⁰

The consequences for labor’s share of income, though, are dire. With gross fac-

¹⁸See also **Figure C.1**.

¹⁹See the discussion in Pitchford (1960) and the subsequent discussions in de La Grandville (1989); Klump and de La Grandville (2000); Klump and Preissler (2000); Palivos and Karagiannis (2010).

²⁰A General Mean of order ξ is defined as $M(\xi) = \left[\sum_{i=1}^n f_i x_i^\xi \right]^{\frac{1}{\xi}}$ where x_i, \dots, x_n are positive numbers (of the same dimension) and where the weights $f_1 \dots f_n$ sum to unity. A General Mean is an increasing function of its order: the mean of order ξ of the positive values x_i with weights f_i is a strictly increasing function in ξ unless all the x_i are equal. This is often known as the de La Grandville hypothesis after de La Grandville (1989). He attributed that the rapid growth in East Asian countries to a high substitution factor elasticity value in their industrial sectors, and their high savings rate. This echoes the argument of Hicks (1932) that larger substitution value entails high transformation rates between sectors of different factor intensity. When one activity contracts to the benefit of another, the production increase in the second sector can be made larger if the substitution elasticity is high.

tor substitutability,²¹ the arguments of the previous section shift into reverse. Capital improvements are capital biased, and the incentives for capital accumulation are accordingly far higher in this regime. Hence the labor share declines with σ (or equivalently ξ).

It should also be emphasized that a BGP does not exist in our model under sufficiently strong factor substitutability. Gross substitutability, as such, implies that Inada conditions at infinity are violated: the marginal product of per-capita capital remains bounded *above* zero as the capital stock goes to infinity. But then there is still the question of whether the lower bound of the marginal productivity of capital, multiplied by the savings rate, is high enough to exceed the capital depreciation rate. If so, and this happens only when the substitution elasticity exceeds a certain threshold $\sigma > \bar{\sigma} > 1$ ²², endogenous growth driven by capital accumulation appears (Jones and Manuelli, 1990; Palivos and Karagiannis, 2010). Combined with the existing growth engine of our model – labor augmenting R&D – both sources of growth then lead to hyper-exponential, explosive growth. Then, even with diminishing returns to factors, capital intensity grows without bounds, labor becomes inessential in production, and hence the capital income share tends to unity. We rule such cases out of our analysis (see **Figure 2–Figure 4**).

4.4 Relation to Piketty’s Laws

As our model endogenizes both economic growth and factor income shares, it constitutes an appropriate framework for studying the two “Fundamental Laws of Capitalism” formulated by Piketty (2014):

- (i) that the degree of capital deepening K/Y goes up whenever the economic growth rate g goes down, and;
- (ii) that the capital share π goes up whenever the growth rate g goes down.

Our setup has the advantage over Piketty’s in that all three variables are endogenous, and hence one can legitimately observe whether changing some parameters implies relations that are in line with Piketty’s claims, or go in the opposite direction.²³

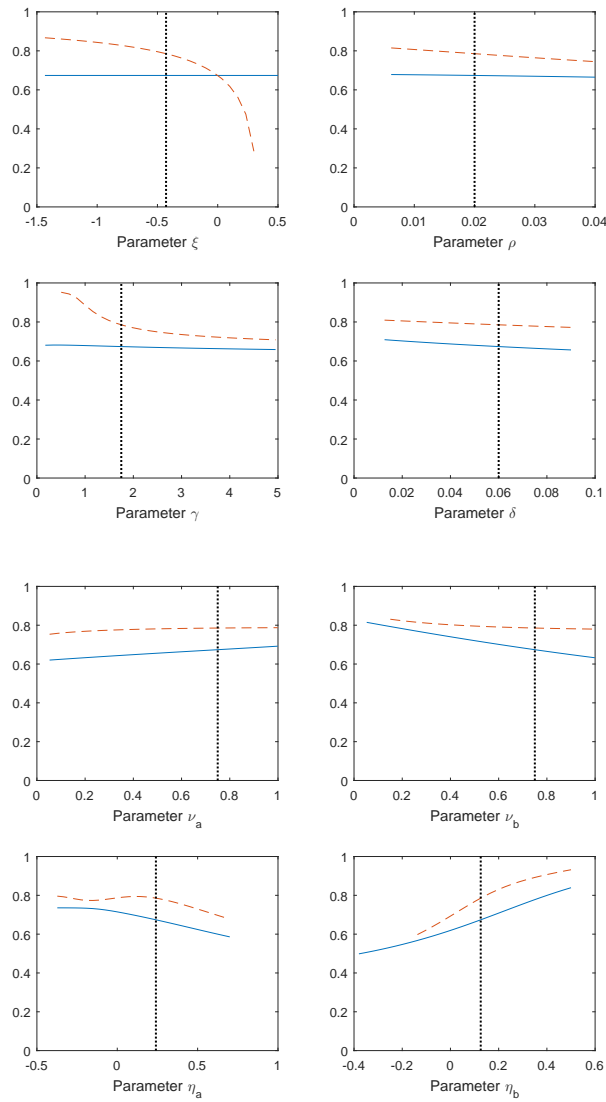
First, taking Piketty’s claims together logically implies that K/Y and the capital share π are positively correlated, suggesting that capital and labor should be gross substitutes ($\sigma > 1$), see e.g., equation (32). This is a widely recognized issue with Piketty’s claims (see e.g. Rognlie (2015); Oberfield and Raval (2018)). In our baseline parametrization, recall, we assume gross complements instead.

²¹In the Cobb–Douglas case of $\xi = 0$, factor shares are constant and at their predetermined sample average. Thus $\pi|_{\xi=0} = \pi_0$.

²²The critical threshold level for the substitution elasticity (to generate such ‘perpetual growth’) can be shown to be increasing in the growth of labor force and decreasing in the saving rate, see de La Grandville (1989).

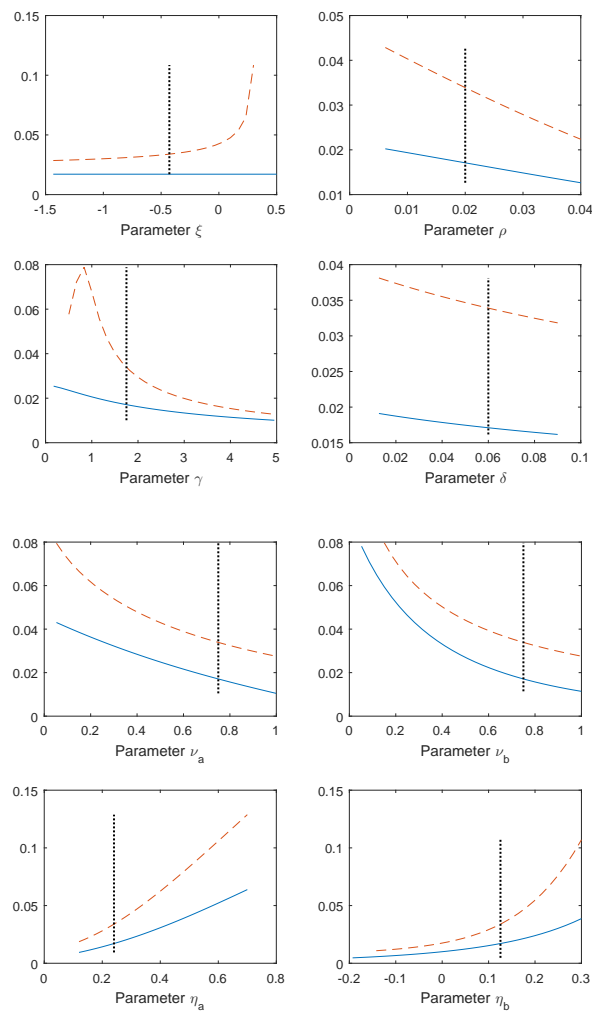
²³The same issue has been recently taken up by Irmen and Tabaković (2018) but our model is relatively better suited to addressing this issue because it departs from the Cobb–Douglas technology.

FIGURE 2: DEPENDENCE OF EQUILIBRIUM LABOR SHARE ON MODEL PARAMETERS.



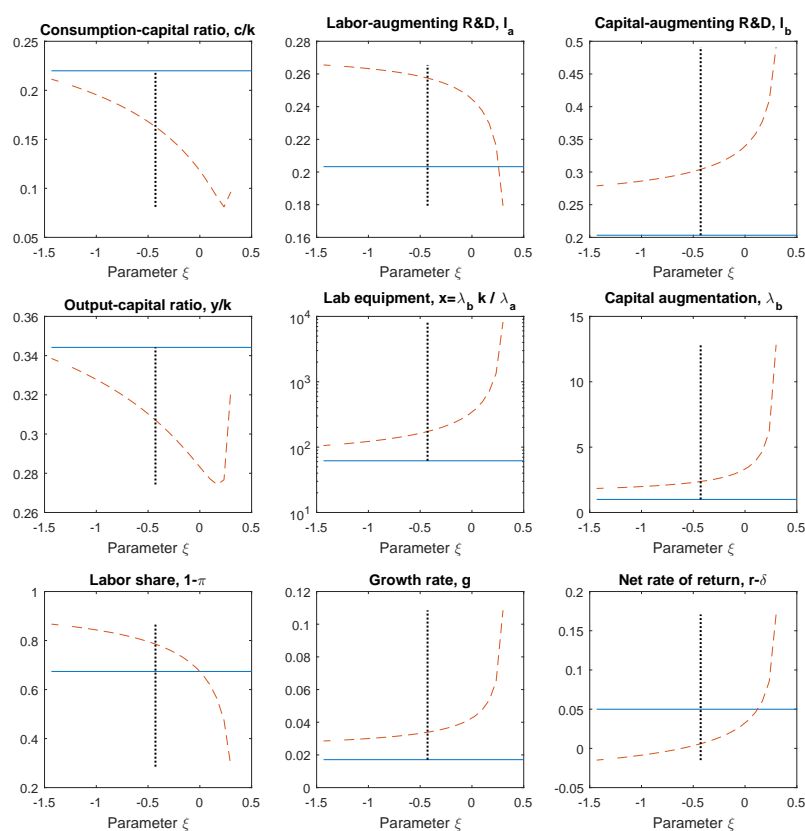
Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. **Social planner allocation** (dashed lines), **decentralized equilibrium** (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

FIGURE 3: DEPENDENCE OF EQUILIBRIUM GROWTH ON MODEL PARAMETERS.



Notes: The real economic growth rate g on vertical axis; corresponding parameter support on the horizontal axis. **Social planner allocation** (dashed lines), **decentralized equilibrium** (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

FIGURE 4: DEPENDENCE OF BALANCED GROWTH PATH ON ELASTICITY OF SUBSTITUTION, DA vs. SP.



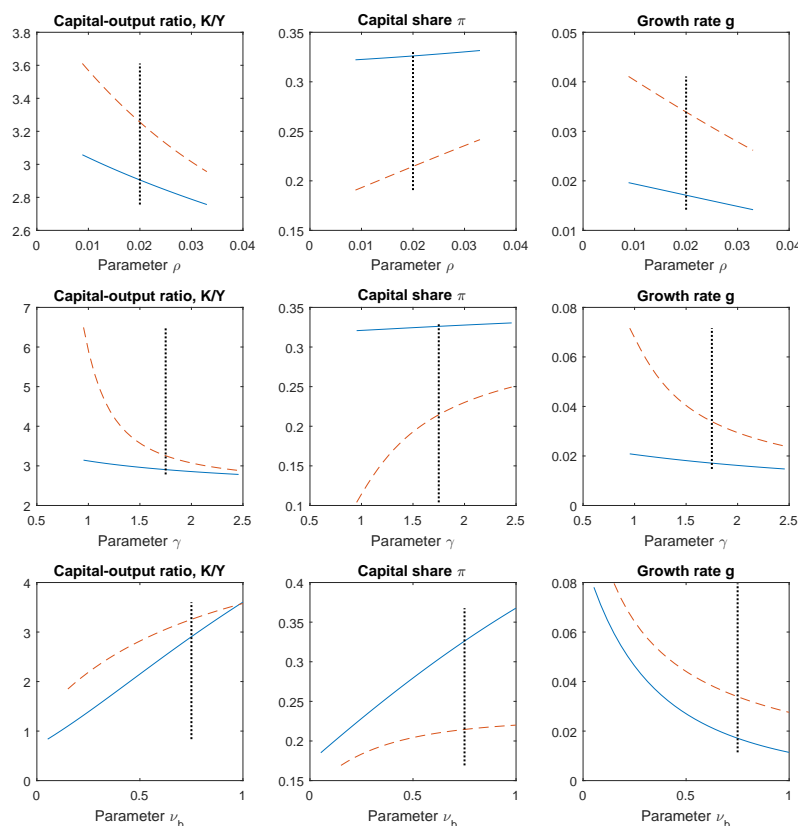
Note: Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

Second, inspection of **Figure 5** reveals that under the baseline calibration, both in the decentralized equilibrium and the social planner allocation:

- when households become more patient (ρ goes down) or more willing to substitute consumption intertemporally (γ goes down), only law (ii) holds: the growth rate g goes up, the K/Y ratio goes up, and the capital share π goes down;
- when the lab equipment exponent ν_b in capital augmenting R&D goes up, both laws are verified: the growth rate g goes down, the K/Y ratio goes up, and the capital share π goes up.

Third, we found (recall **Figure 4**) that as the elasticity of substitution goes up, the optimal growth rate g goes up hand in hand with the capital share π and the K/Y ratio. In such case, *both* of Piketty’s laws are violated.

FIGURE 5: DEPENDENCE OF BALANCED GROWTH PATH ON ρ , γ AND ν_b , DA VS. SP.



Note: Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

4.5 Is the Decentralized Economy Characterized by Excessive Volatility?

In the data, we know that – irrespective of the concept utilized – labor shares are highly persistent and variable.²⁴ Although bounded within the unit interval and theoretically stationary, in the data labor income shares often appear to be characterized by marked volatility and long swings. In particular, around 80% of total labor share volatility in the US (1929–2015) has been due to fluctuations in medium-to-long run frequencies (beyond the 8 year mark). As opposed to the short-run component of the labor share, its medium-to-long run component has also been procyclical (Growiec et al., 2018).

Other than undermining the case for aggregate Cobb–Douglas production, such protracted swings and volatilities also raise the question of whether our framework can generate and rationalize these long cycles. Growiec et al. (2018) have confirmed this conjecture for the decentralized allocation of the current model. The question is however equally interesting for the social planner case. Are cycles in factor income shares socially optimal? If so, then, for instance, (stabilization) policies to mitigate

²⁴For international evidence see Jalava et al. (2006); Bengtsson (2014); Mućk et al. (2018).

labor share or real volatility might be appraised differently.

Table 3 makes the relevant comparisons across our maintained cases. It shows that the decentralized allocation features relatively shorter cycles but also faster convergence to the BGP. Hence, it cannot be unambiguously claimed directly that the decentralized equilibrium has *excessive volatility*. If both allocations were to start from the same initial point outside of the BGP then the decentralized allocation would exhibit a greater frequency but smaller amplitude of cyclical variation.

TABLE 3: DYNAMICS AROUND THE BGP

	Baseline		C–D		Piketty	
	$\xi = -0.43$		$\xi = 0$		$\xi = 0.2$	
Allocation	DA	SP	DA	SP	DA	SP
Pace of convergence* (% per year)	6.3%	4.2%	5.8%	3.7%	5.2%	2.9%
Length of full cycle [†] (years)	52.6	76.7	79.8	83.2	144.0	100.3
Labor Share Cyclical	+	+	0	0	–	–
Amplitude of $1 - \pi$ relative to y/k	62.0%	48.0%	N/A	N/A	28.0%	44.0%

Note: * computed as $1 - e^{rr}$ where $rr < 0$ is the real part of the largest stable root; [†] computed as $2\pi/ir$ where $ir > 0$ is the imaginary part of two conjugate stable roots (if they exist). “N/A” denotes not available/applicable.

Having scrutinized the robustness of this dynamic result by extensively altering the model parametrization, we conclude that while the decentralized equilibrium generally exhibits shorter cycles, the ordering of both allocations in terms of the pace of convergence can sometimes be reversed. This finding lends partial support to the claim that the decentralized equilibrium is perhaps likely to feature greater labor share volatility compared to the social optimum. However, it is worthwhile to point out that oscillations in the labor income share can still be socially optimal in this model. In our model, the reasons why we have this cycle around the BGP is the following. It reflects the tension between the two R&D sectors in capital and labor: acceleration in each of them has conflicting impacts on the labor share.

Moreover, we also obtain quantitative predictions on the cyclical co-movement of the original model variables (including the economic growth rate g and the labor share $1 - \pi$).²⁵ It turns out, both for the decentralized and optimal allocation, that all variables except for the consumption-capital ratio $u = c/k$ oscillate when converging to the steady state, with the same frequency of oscillations. The level of capital augmenting technology λ_b , the “lab equipment” term x , and labor augmenting R&D employment ℓ_a are always counter-cyclical, employment in production ℓ_Y is always pro-cyclical, whereas the cyclicality of capital augmenting R&D ℓ_b is ambiguous (in the baseline calibration, ℓ_b is pro-cyclical in the decentralized allocation but coun-

²⁵This is done by inspecting the eigenvector associated with the largest stable root of the Jacobian of the system at the steady state.

tercyclical in optimal one). Furthermore, as long as capital and labor are gross complements, the labor income share $1 - \pi$ is unambiguously pro-cyclical as well. These features of cyclical co-movement align well with the empirical evidence for the US medium-term cycle. In particular, the US labor share is indeed procyclical over the medium-to-long run – despite its counter-cyclicity along the business cycle.

5 Conclusions

Endogenous growth theory tends to suggest that the socially optimal level of activity dominates the decentralized outcome. We confirmed this using a micro-founded, calibrated two-sector R&D endogenous growth model. Due to the existence of imperfect competition, externalities and duplication possibilities in production, the decentralized allocation produces a socially suboptimal level of R&D, and particularly too little labor augmenting R&D. This, in addition to a suboptimal level of capital accumulation, translates into too few efficient units of capital $\lambda_b \cdot k/y$ at the BGP and thus too low equilibrium growth.

But what of the labor share? Despite its importance, the conclusions for this variable have not yet been drawn in the literature. Our objective was to bridge that knowledge gap. We found that the same components which drive a wedge between the social and decentralized growth rate, reappear in the labor share analysis: markets fail to account for the external effects of physical capital on R&D activity; the shadow price of capital in the SP is replaced by its market price which accounts for imperfectly competitive mark-ups; markets fail to internalize the labor augmenting R&D duplication effects, and fail to account for the external effects of accumulating knowledge on future R&D. Given these frictions, we found that if the elasticity of factor substitution is below unity (as the bulk of evidence suggests for the US), then the decentralized labor share is indeed socially suboptimal. This difference, moreover, is large, around 17%.

Effectively, the only parameter which reverses this ordering is the elasticity of substitution. Our finding is therefore a very strong and robust result. However the case of gross substitutes tends to yield highly counter-factual outcomes otherwise, rendering it (in the context of this model) a somewhat unrealistic benchmark. For example an elasticity of $\sigma = 1.25$ (i.e., only marginally above Cobb Douglas), does produce a decentralized labor share above that of the social planner, but then the latter is as low as 0.52. As a simple point of comparison, according to the ILO definition of the labor share (using annual data from 1960 to the present), no G7 country has fallen below a labor share of 0.5. Moreover such a mild perturbation away from Cobb–Douglas already produces equilibrium per capital growth rates of around 6%. Thus, the mantra that “growth is always good for labor” is conditional on which technological frictions one models, and what is the degree of factor substitution in the economy.

A final interesting finding is that both the decentralized and social optimum are characterized by cycles. At the high frequency end this covers technology adoption costs, but in the longer run it reflects the tension between different R&D sectors and their effect on factor shares. There is some support for the claim that the decentralized

equilibrium is likely to feature greater labor share volatility compared to the social optimum. Analysis of the model, and the cycles contained therein, moreover found no evidence for Piketty’s “two laws of Capitalism”.

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SUPPLEMENTARY MATERIAL

A Steady State of the Transformed System

A.1 Social Planner Allocation

The steady state of the transformed dynamical system implied by the social planner solution satisfies:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a} \quad (\text{A.1})$$

$$\gamma g + \delta + \rho = z \left(\pi + \frac{1 - \pi}{\ell_Y} \left(\frac{\eta_a \ell_a}{\nu_a} + \frac{\eta_b \ell_b}{\nu_b} \right) \right) \quad (\text{A.2})$$

$$g = z - \zeta \frac{\dot{a}}{k} - u - (\delta + n) \quad (\text{A.3})$$

$$d = B (\lambda_b^{-\omega} x^{\eta_b} \ell_b^{\nu_b}) \quad (\text{A.4})$$

$$(1 - \gamma)g + n - \rho = d \left(\frac{\pi}{1 - \pi} \frac{\ell_Y \nu_b}{\ell_b} + (\phi + \eta_a) \frac{\nu_b \ell_a}{\nu_a \ell_b} - \omega + \eta_b \right) \quad (\text{A.5})$$

$$(1 - \gamma)g + n - \rho = -g \left(\frac{\ell_Y \nu_a}{\ell_a} - \eta_a - \eta_b \frac{\ell_b \nu_a}{\ell_a \nu_b} \right) \quad (\text{A.6})$$

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b}{\lambda_{b0}} \right)^\xi \left(\frac{z}{z_0} \right)^{-\xi} \quad (\text{A.7})$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1 - \pi_0) \left(\frac{x_0}{x} \frac{\ell_Y}{\ell_{Y0}} \right)^\xi \right)^{1/\xi} \quad (\text{A.8})$$

This non-linear system of equations is solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the social planner allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22)–(24), (2) and (5), taking the BGP equality (21) as given.

A.2 The Decentralized Allocation

As the case of the social planner, the Euler equations and dynamics of state variables are rewritten in terms of *stationary* variables. The steady state of the transformed system satisfies:

$$g = \hat{\lambda}_a = \hat{k} = \hat{c} = \hat{y} = A(\lambda_b^*)^\phi (x^*)^{\eta_a} (\ell_a^*)^{\nu_a} \quad (\text{A.9})$$

$$\gamma g + \rho = r - \delta \quad (\text{A.10})$$

$$g = z - \zeta \frac{\dot{a}}{k} - u - (\delta + n) \quad (\text{A.11})$$

$$d = B (\lambda_b^{-\omega} x^{\eta_b} \ell_b^{\nu_b}) \quad (\text{A.12})$$

$$g \frac{\ell_Y}{\ell_a} = r - \delta \quad (\text{A.13})$$

$$g = r - \delta + d \left(1 - \frac{\pi}{1 - \pi} \frac{\ell_Y}{\ell_b} \right) \quad (\text{A.14})$$

$$r = \varepsilon \pi z \quad (\text{A.15})$$

$$\frac{\pi}{\pi_0} = \left(\frac{\lambda_b}{\lambda_{b0}} \right)^\xi \left(\frac{z}{z_0} \right)^{-\xi} \quad (\text{A.16})$$

$$\frac{z}{z_0} = \frac{\lambda_b}{\lambda_{b0}} \left(\pi_0 + (1 - \pi_0) \left(\frac{x_0}{x} \frac{\ell_Y}{\ell_{Y0}} \right)^\xi \right)^{1/\xi} \quad (\text{A.17})$$

This non-linear system of equations is solved numerically, yielding the unique steady state of the de-trended system, and thus the unique BGP of the model in original variables. All further analysis of the decentralized allocation is based on the (numerical) linearization of the 5-dimensional dynamical system of equations (22')–(24'), (2) and (5), taking the BGP equality (31) as given.

B Additional Calibrated Parameters

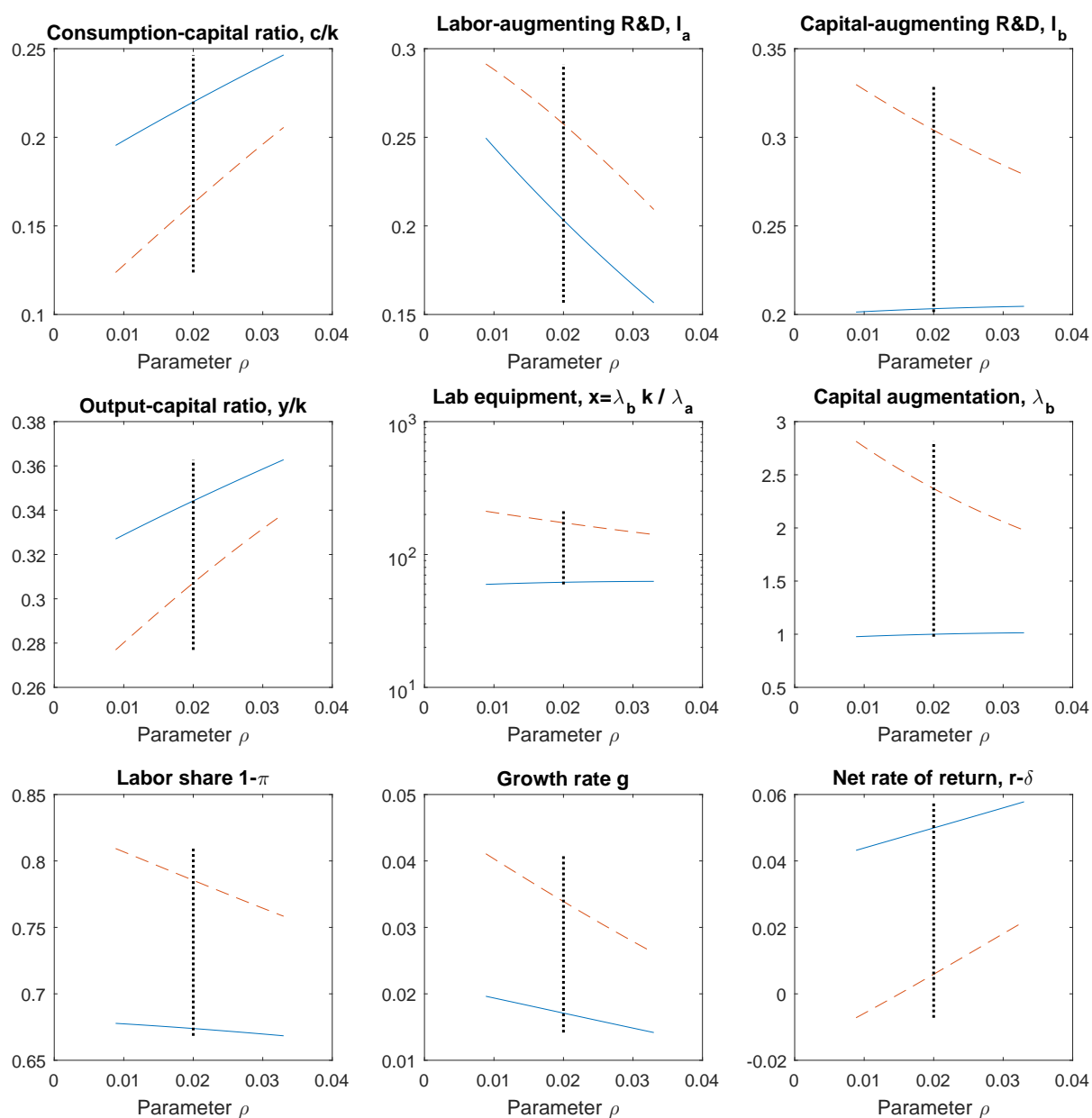
We solve the four remaining equations in system (A.9)–(A.17) with respect to the remaining parameters, see **Table B.1**. All these parameters are within admissible ranges. For instance, Pessoa (2005) estimates values for the obsolescence parameter between 0-15%; our endogenously determined value is thus centered in that range. Comparing $\eta_a = 0.24$ with $\eta_b = 0.13$ signifies that, first of all, lab equipment (effective capital augmentation of the R&D process) assuredly matters for R&D productivity, and second, that it is relatively more important for inventing new labor augmenting technologies than capital augmenting ones. Moreover, with $\phi = 0.3$, labor augmenting R&D – the ultimate engine of long-run growth – is substantially reinforced by spillovers coming from the capital augmenting R&D sector. On the other hand, $\omega = 0.5$ means that the scope for capital augmenting R&D is quite strongly limited by decreasing returns. Given this benchmark calibration, as we said in the main text, the steady state is a saddle point.

TABLE B.1: BASELINE CALIBRATION: ADDITIONAL PARAMETERS

Parameter	Value	
<i>Labor augmenting R&D</i>		
Unit productivity	A	0.02
Lab equipment exponent	η_a	0.24
<i>Capital Augmenting R&D</i>		
Unit productivity	B	0.16
Lab equipment exponent	η_b	0.13
Degree of decreasing returns	ω	0.50
Obsolescence rate	d	0.08
Spillover from capital to labor augmenting tech. change	ϕ	0.30
Technology choice externality	ζ	115.28

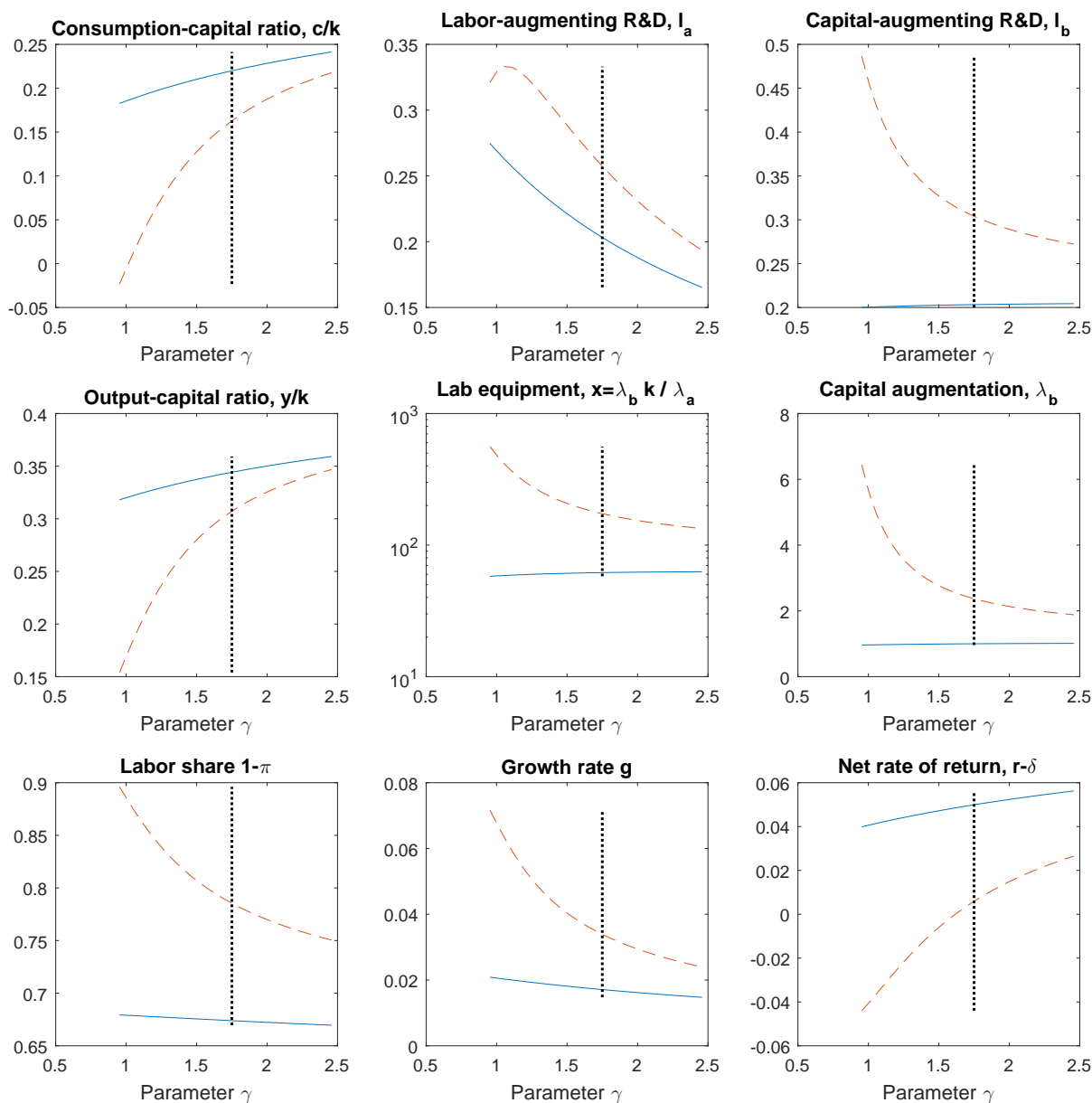
C Additional Figures

FIGURE C.1: COMPARING BALANCED GROWTH PATHS, DA VS. SP. DEPENDENCE ON THE TIME PREFERENCE.



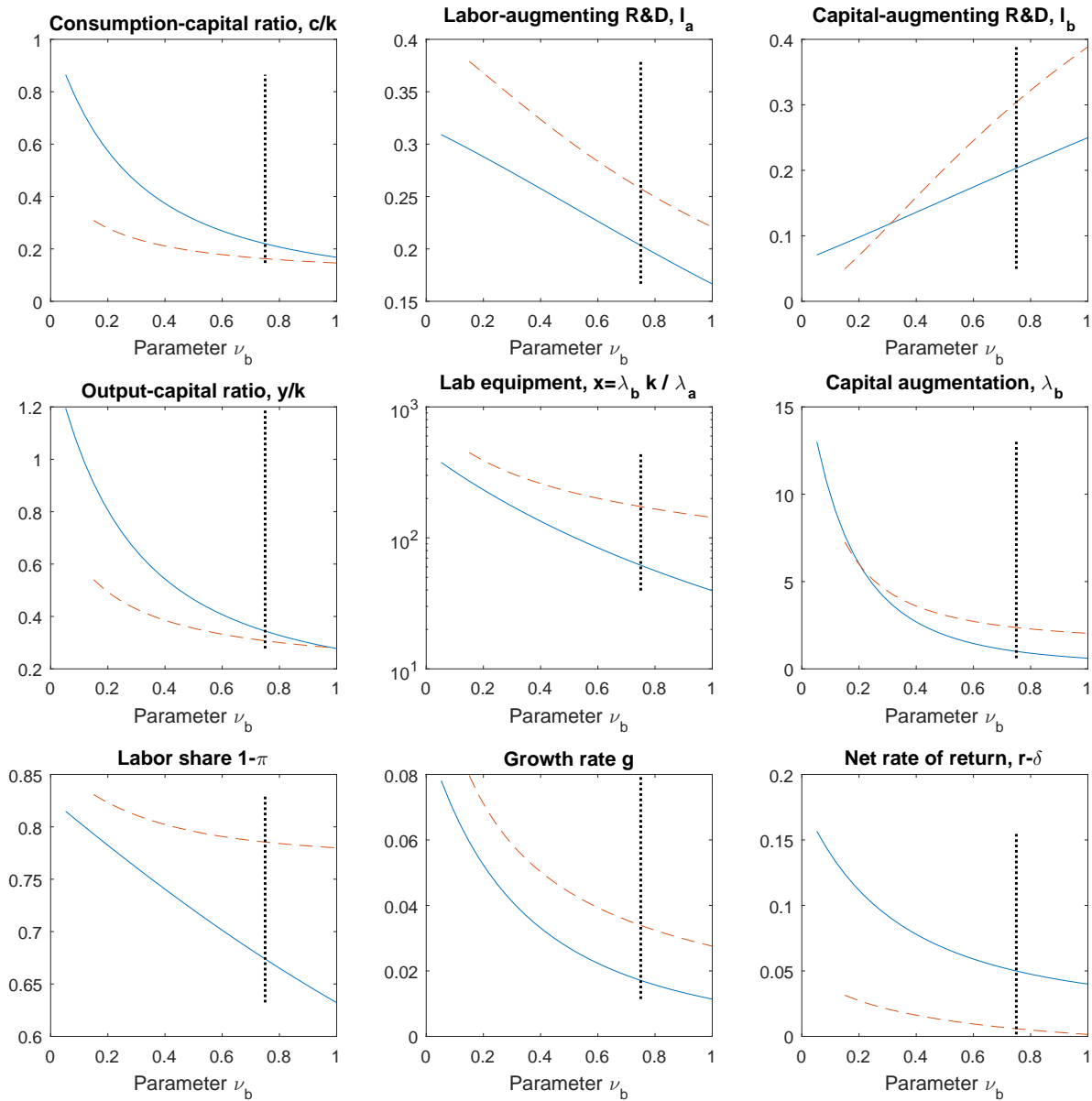
Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. **Social planner allocation** (dashed lines), **decentralized equilibrium** (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

**FIGURE C.2: COMPARING BALANCED GROWTH PATHS, DA vs. SP.
DEPENDENCE ON THE INTERTEMPORAL ELASTICITY OF SUBSTITUTION IN CONSUMPTION.**



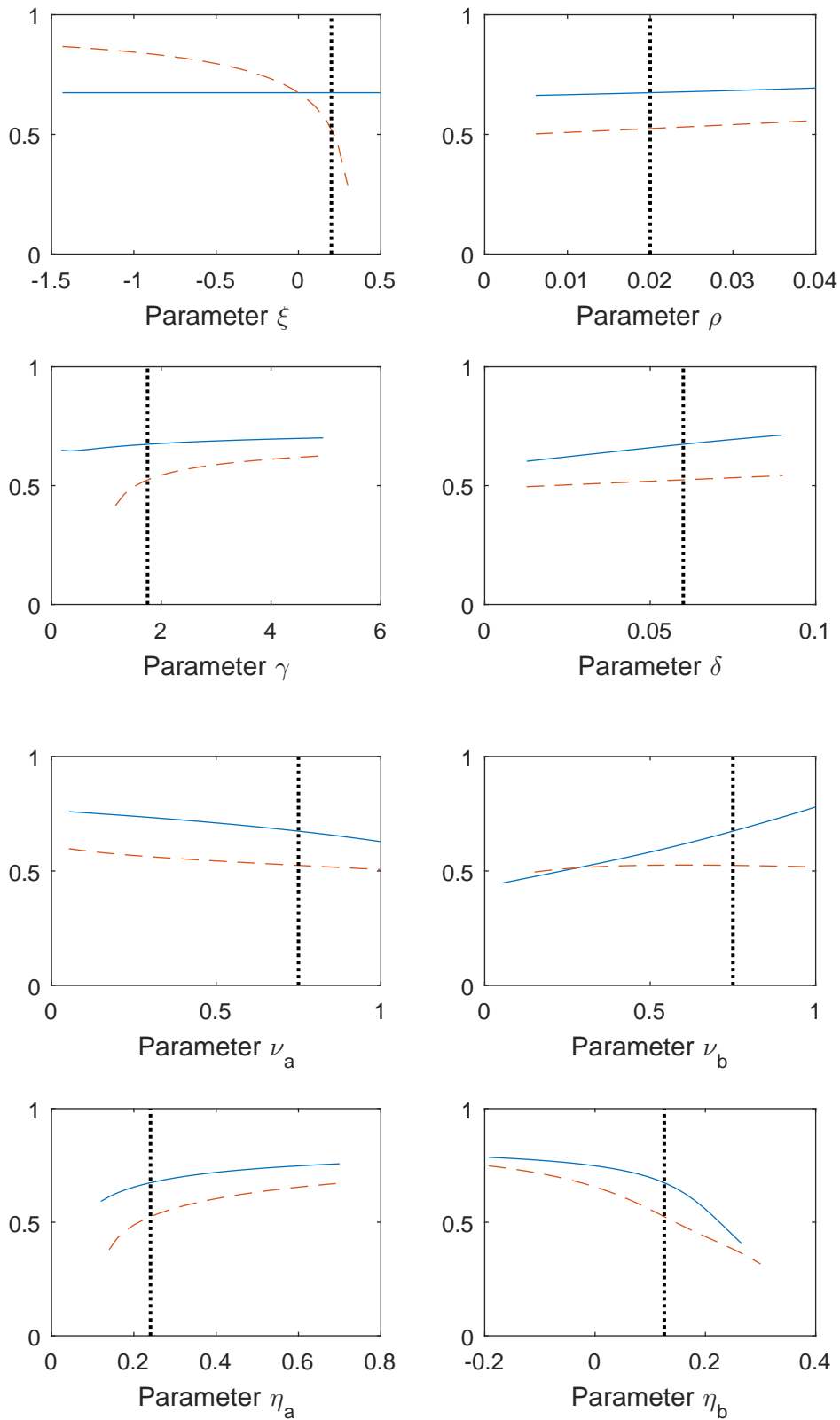
Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. **Social planner allocation** (dashed lines), **decentralized equilibrium** (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

FIGURE C.3: COMPARING BALANCED GROWTH PATHS, DA VS. SP. DEPENDENCE ON THE LAB EQUIPMENT EXPONENT IN CAPITAL AUGMENTING R&D.



Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. **Social planner allocation** (dashed lines), **decentralized equilibrium** (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

FIGURE C.4: DEPENDENCE OF EQUILIBRIUM LABOR SHARE ON MODEL PARAMETERS, FOR THE ALTERNATIVE CALIBRATION OF $\sigma = 1.25$ ($\xi = 0.2$).



Notes: $1 - \pi$ on vertical axis; corresponding parameter support on the horizontal axis. Social planner allocation (dashed lines), decentralized equilibrium (solid lines). The vertical dotted line in each graph represents the baseline calibrated parameter value.

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