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Pro-cyclical emissions, real externalities, and optimal monetary policy[☆]

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ABSTRACT

We study optimal monetary policy in an analytically tractable New Keynesian DSGE-model with an emission externality. Empirically, emissions are strongly pro-cyclical and output in the flexible price equilibrium overreacts to productivity shocks, relative to the efficient allocation. At the same time, output under-reacts relative to the flexible price allocation due to sticky prices. Therefore, it is not optimal to simultaneously stabilize inflation and to close the natural output gap, even though this would be feasible. Real externalities affect the LQ-approximation to optimal monetary policy and we extend the analysis of Benigno and Woodford (2005) to inefficient flexible price equilibria. For central banks with a dual mandate, optimal monetary policy places a larger weight on output stabilization and targets a non-zero natural output gap, implying a higher optimal inflation volatility.

1. Introduction

There is a broad consensus that greenhouse gas emissions inflict severe damages on the wider economy through their contribution to climate change. Economic theory suggests that Pigouvian emission taxes are the best instrument to address such an externality and it is becoming increasingly clear that central banks can play at most a supporting role in addressing externalities related to emissions. The short term nominal interest rate is naturally not well-suited to address long term issues such as climate change and even the unconventional central bank toolkit provides very limited potential to induce a sectoral re-allocation away from fossil fuels.[1] However, it remains an unanswered question how monetary policy should adapt to a world characterized by climate change.

In this paper, we focus on the pro-cyclicality of emissions at business cycle frequencies, its interplay with nominal rigidities and the implications for optimal monetary policy in an analytically tractable environmental New Keynesian framework. Fig. 1 demonstrates that carbon dioxide emissions are highly pro-cyclical in both advanced and emerging economies, and at a global

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[1] We refer to Giovanardi et al. (2023) for an assessment of preferential collateral haircuts for green bonds and to Ferrari and Nispi Landi (2023) for green QE.

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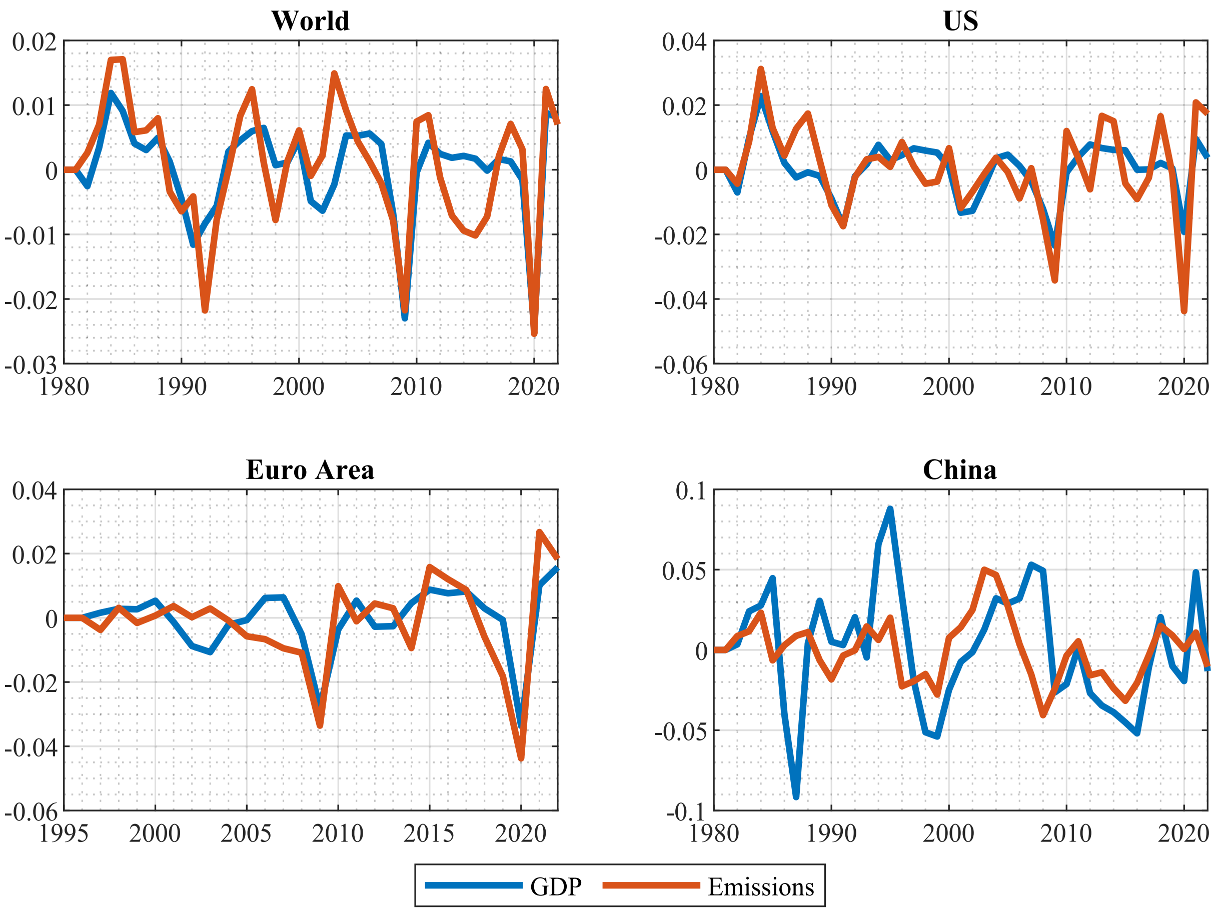

Fig. 1. Pro-Cyclical Carbon Dioxide Emissions. This figure shows real GDP and carbon dioxide emissions over time. Data are annual and de-trended using a one-sided HP-filter with smoothing parameter 6.25. The full-sample correlations are 0.77 for the World, 0.78 for the US, 0.78 for the Euro Area and 0.31 for China.

level. The correlation between (de-trended) emissions and (de-trended) GDP is close to 80% at the global level and it is remarkably similar between the US and the Euro Area. Doda (2014) and Khan et al. (2019) provide earlier evidence. Pro-cyclical emissions are macroeconomically relevant. For example, a two percent expansion of global emissions above trend would inflict damages worth one trillion USD at recent estimates of the social cost of carbon (Bilal and Kaenzig, 2024). This corresponds to around one percent of global GDP.[2] In welfare terms, households would gain 0.41% in consumption equivalents if emissions were constant rather than pro-cyclical.

With pro-cyclical emissions, a Pigouvian emission tax does not implement the efficient allocation if it only addresses the emission externality in the long run but does not respond to business cycle fluctuations.[3] Individual firms do not take into account that expanding their production in response to a productivity shock inflicts damages on the wider economy, such that output in the competitive equilibrium overreacts relative to the efficient allocation. The discrepancy between flexible price and efficient allocation is substantial. Under plausible calibrations of emission damages, a productivity shock that increases output by one percent in competitive equilibrium implies an optimal output expansion by 0.9% in the planner solution. The difference of 0.1% is of similar

[2] These numbers are computed based on emissions and GDP data from 2022, using a social cost of carbon of 1.367 USD/tCO2. A two percent increase in US emissions above trend would induce damages of 0.15% of global GDP, while the same expansion in the Euro Area or China would correspond to damages of 0.05% and 0.3% of global GDP, respectively.

[3] An optimal pro-cyclical emission tax that addresses the over-reaction of economic activity to productivity shocks in the competitive equilibrium is a well documented feature of the canonical environmental RBC model, going back to Heutel (2012). More recently, Muller (2023) demonstrates that the pro-cyclicality of emissions is an important driver of the "green interest rate", which adjusts the concept of the natural interest rate for environmental damages. Similarly, Benmir et al. (2024) argue that household stochastic discount factors that are consistent with macro-finance moments can only be reconciled with a strongly pro-cyclical social cost of carbon.

magnitude as forecast errors for output growth and inflation in advanced economies and, thus, also a practically relevant concern for monetary policy.[4]

The relative overreaction of output in the flexible price equilibrium interacts non-trivially with nominal rigidities. Consider a positive shock to total factor productivity. A price setting friction (Calvo, 1983) prevents a fraction of firms from reducing prices, such that the economy expands by less than it would do under flexible prices. Under conventional parameters for the nominal rigidity, the discrepancy between sticky price and efficient output is approximately the same as the discrepancy between flexible price and efficient output, underscoring the first-order importance of pro-cyclical emissions for macroeconomic stabilization policies.

In the standard New Keynesian model without emission externalities, the central bank aims at closing the gap between the sticky price and flexible price output. We refer to this as the *natural* output gap. By contrast, with pro-cyclical emissions, closing the natural output gap does not implement the efficient allocation. What matters from a normative perspective is the difference between the output reaction under sticky prices and the output reaction in the efficient allocation. We refer to this as the *efficient* output gap. Price stickiness attenuates the overreaction of the competitive equilibrium vis-a-vis the efficient allocation. Therefore, the efficient output gap is ambiguously affected by the emission externality. While it is generally larger (less negative) than the natural output gap, it can even become positive. This implies that, in contrast to the standard New Keynesian model, the central bank is unable to perfectly stabilize both inflation and the efficient output gap. Equivalently, the central bank could perfectly stabilize inflation and natural output gap, but this is not optimal. Divine coincidence as defined by Blanchard and Gali (2007) is broken.[5]

We obtain this insight in an analytically tractable environmental New Keynesian model that combines elements from monetary policy theory (Clarida et al., 1999; Woodford, 2011) with a simple representation of socially harmful emissions in the spirit of the workhorse environmental RBC model (Heutel, 2012). The competitive equilibrium of this model is characterized by a dynamic IS equation, the New Keynesian Phillips curve, a monetary policy equation linking the nominal interest rate to macroeconomic variables such as inflation, and the law of motion for the stock of atmospheric carbon dioxide, which inflicts damages on economic activity. As a preliminary step, we show that we can (quantitatively) approximate such a four-equation model by a three-equation model in which emission damages depend on the flow of emissions. This is justified because the *over-reaction* of output in the flexible price equilibrium relative to the efficient allocation is very similar in both models. The reason is that, in the model with persistent emissions, damages increase slightly but persistently after an expansionary shock, while they increase strongly but only briefly in the three-equation model. As the social planner is concerned with the present value of emission damages, the discrepancy between efficient and natural output reaction to a productivity shock is approximately independent of the degree of emission persistence. Eliminating the law of motion for atmospheric carbon dioxide allows us to analyze monetary policy without tracking the stock of emissions as an additional endogenous state variable.

As our main contribution, we present an analytical characterization of optimal monetary policy in the three-equation model along the lines of Clarida et al. (1999) and Woodford (2011). Our analysis is applicable for central banks with a dual mandate. Conceptually, optimal monetary policy under cyclical emissions is a second best solution to a utilitarian welfare-maximization problem. With a time-invariant tax, optimal monetary policy addresses two dynamic frictions with only one instrument - the nominal interest rate - and will not be able to offset both inefficiencies at once. If appropriate cyclical adjustments to emission taxes were in place, optimal monetary policy could be conducted as usual.[6] The presence of such a second dynamic inefficiency affects the central bank's objective function, which is derived from first principles.

Our key methodological innovation lies in the derivation of the central bank loss function when the competitive equilibrium is (dynamically) inefficient. As customary in the literature, the loss function builds on a second order approximation to the household utility function and uses equilibrium conditions to express this in terms of output gap and inflation. It turns out to be essential to include an additional condition that takes the effect of economic activity on emission damages into account. Without this additional condition, the central bank would simply take emission damages as given, so that the resulting loss function would inherit the market failure from the flexible price equilibrium and prescribe to close inflation and natural output gap in all states, which is clearly not socially optimal. The additional expression, which ensures that the loss function actually internalizes the emission externality, introduces linear terms related to the *level* of emission damages into the loss function. In order to cleanly separate business cycle stabilization objectives from steady state inefficiencies, we eliminate those terms by a second order approximation of the relationship between economic activity and emission damages. This step is in the spirit of Benigno and Woodford (2005), who use a second order approximation of the Phillips curve for monopolistic distortions in steady state.[7]

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[4] Andersen (1997) reports median absolute forecast errors for (same-year) GDP growth between 0.5 and 0.9%, and 0.3% and 0.7% for (same-year) inflation, see also D'Andrea et al. (2025) for more recent evidence from US inflation forecast errors. Gatti et al. (2024) report slightly larger growth forecast errors for emerging economies.

[5] Breaking divine coincidence in the presence of productivity shocks requires frictions that go beyond nominal rigidities. For example, Faia (2009) shows that search frictions on the labor market render the flexible price allocation infeasible. In contrast, the flexible price allocation is implementable in our framework, but it is not optimal to do so. Adao et al. (2003) demonstrate that in an economy with cash-in-advance constraints, it is not optimal to fully stabilize prices and output gaps, which is conceptually similar to our results. Sims et al. (2023) discuss the role of financial shocks as inflation shifters in the New Keynesian Phillips curve, which also break divine coincidence.

[6] It appears rather implausible from an institutional background that central banks can directly address pro-cyclical emissions, for example by purchasing and selling emission permits. Arguably, climate policy is usually conducted by a fiscal authority over longer run horizons, while overreactions of output to business cycle fluctuations typically belong to the domain of macroeconomic stabilization policies.

[7] We also approximate the Phillips curve up to second order in order to allow for large steady state discrepancies between efficient allocation and flexible price equilibrium, for example due to the absence of carbon taxes.

The resulting loss function differs from the standard New Keynesian model in two ways. First, it features a non-zero target level for the natural output gap that reflects the over-reaction of output in competitive equilibrium. The target level is negative for a positive TFP shock and increasing in absolute terms in the emission externality. Intuitively, the degree of over-production increases in the externality, such that the deviation from the flexible price equilibrium also increases in the externality. Second, optimal monetary policy places a higher weight on output stabilization if the externality is more severe, as larger emission damages only affect the welfare losses from large output gaps, but leave the severity of nominal rigidities unaffected. This result resonates with Clarida et al. (1999), who show that the weight on inflation stabilization is higher if the output is *below* its efficient level due to monopolistic distortions. Here the opposite intuition applies, since flexible price output is *above* the efficient level.

Equipped with the central bank loss function, the LQ-approach to optimal monetary policy under discretion then simply minimizes the loss function subject to the (steepened) New Keynesian Phillips curve. In addition to the discrepancy between natural and efficient output response to productivity shocks discussed above, the emission externality affects the competitive equilibrium by steepening the New Keynesian Phillips curve. Intuitively, emission damages endogenously dampen productivity, such that natural output, actual output and the natural output *gap* respond less to a productivity shock. At the same time, emission damages do not affect firms' ability to change prices, such that the natural output gap moves less for a given change in inflation. The solution to the LQ-problem shows that the optimal natural output gap is a linear combination between its target level and zero, i.e. the flexible price allocation. The (optimal) natural output gap is closer to its target level if the emission externality is large, both through its effect on the weight and the slope of the New Keynesian Phillips curve. To dampen the output reaction, the central bank cuts interest rates by less in absolute terms in response to a positive TFP shock than it would do absent emission damages. Consequently, the central bank allows for some dis-inflation after a positive productivity shock. Optimal inflation volatility is larger in the presence of (pro-cyclical) emission externalities.

We also consider the cases of demand and cost push shocks, where in particular the latter have received attention in the context of climate change and monetary policy (Fornaro et al., 2024). The analytical results are obtained by taking analogous steps to the productivity shock case and yield two important insights. First, demand shocks do not affect firms' marginal cost, such that they do not enter the New Keynesian Phillips curve or the central bank loss function. It is optimal and feasible to fully stabilize inflation and natural output gap. Second, cost push shocks already imply that the natural output response differs from the efficient response, even without the externality. While this poses a well-studied trade-off between output gap and inflation stabilization, this trade-off is not directly affected by emission externalities, because there is no direct relationship between economic damages and firms' ability to sell goods at a markup. This is very different for productivity shocks, where the emission externality endogenously dampens expansionary productivity shocks and introduces the discrepancy between welfare-relevant and natural output gap in the first place. Consequently, optimal monetary policy under cost push shocks is merely affected by the steepened New Keynesian Phillips curve, which is relevant from a practical point of view but does not conceptually complicate the conduct of monetary policy.

By providing a simple analytical framework, our analysis contributes to the understanding of welfare-relevant output gaps. This is also relevant for macroeconomic stabilization policies more generally, as these often depend on natural output gaps that need not be efficient.[8] In the spirit of Blanchard and Gali (2007), we show how optimal monetary policy is affected by externalities originating in the real sector. While real externalities have no direct effect on nominal rigidities, they interact with them in a such a way that it is not optimal to implement the flexible price allocation, even though this would be feasible. This analytical insight is more generally applicable to the pro-cyclical depletion of renewable resources, biodiversity degradation and real externalities beyond the environment, such as congestion, that are positively correlated with economic activity at business cycle frequencies.

Taking a quantitative perspective, we find that, in response to a positive one standard deviation TFP shock, the optimal interest rate cut is given by 34 basis points while the natural interest rate declines 40 basis points. This adjustment of 6 basis points is also remarkably similar in the three-equation model, which directly follows from the similarly sized efficient output gap in both models. Consequently, we observe a positive time series volatility of inflation and the efficient output gap in the range of 0.05%. Furthermore, as it is not optimal to stabilize inflation and output gap, there are potential gains from commitment, which do not arise for productivity shocks in the standard New Keynesian model (Clarida et al., 1999). However, the gains from commitment turn out to be quantitatively negligible. Lastly, the welfare gain of optimal monetary policy is smaller (0.05% in consumption equivalents) than in a counterfactual economy without emissions (0.07% in consumption equivalents), as the central bank addresses two dynamic inefficiencies with one instrument at the same time.

The size of the interest rate adjustment term in response to the shocks and the welfare gain of optimal monetary policy are quite robust to reasonable parameter variations in the macro and climate-related parts of the model. It is slightly smaller but still relevant if prices are less sticky, while it becomes larger if household risk aversion is increased, as closing the efficient output gap becomes more important in this case. The most important parameters concern the climate block. If the climate externality is less severe or if we add rest-of-the-world emissions, optimal monetary policy is closest to the standard New Keynesian model. Across parameter variations, the welfare gain of optimal monetary policy lies in the range of 0.04%–0.13% in consumption equivalents. The policy rate adjustment term ranges from 3 to 11 basis points across parameter variations and is smallest in the extension with rest-of-the-world emissions. The effect size appears reasonable, as it is not large but still relevant given that we consider a rather small shock and that the typical interest rate notch is 25 basis points.

[8] Our analysis also relates to the literature of optimal monetary policy in the presence of hysteresis effects. If such effects are present, it is not optimal to close the natural gap. In sharp contrast to a setting with negative real externalities, however, optimal monetary policy in the presence of hysteresis is *more* expansionary in response to a positive TFP shock, see Cerra et al. (2023) and the references therein.

We want to stress that the purpose of the quantitative analysis is to show that our analytical results are empirically relevant, rather than to provide country-specific policy recommendations. For a more specific analysis, some remarks on the institutional setting are in order. In the European Union, for instance, climate policy is implemented by a combination of the European emission trading system (ETS) and national emission taxes, where coverage-weighted emission prices and taxes are of a very similar magnitude (Kaenzig and Konradt, 2024). The differences between price-based (emission tax) and quantity-based (ETS) regulation are subject to a large body of literature in public economics. This goes back to the seminal work of Weitzman (1974), who derives equivalence conditions between price- and quantity-based regulation. Applying this to a New Keynesian DSGE setting, Annicchiarico and Di Dio (2015) and Annicchiarico and Di Dio (2017) numerically study different environmental policy regimes and show that the emission taxes and ETS give rise to different cyclical properties.

How this distinction affects the pro-cyclicality of emissions, which is ultimately relevant for monetary policy, crucially depends on the emission trading system and on the assumptions on technological choice in the firm sector ("emission abatement"). The simple environmental New Keynesian model that we study abstracts from technological choices so that one unit of production always entails one unit of emissions. If emission permits can only be used in the period of issuance, the optimal carbon tax and the optimal permit price coincide in this model. Consequently, issuing a pro-cyclical amount of emission permits implements the efficient allocation.[9] It follows that an emission cap which does not respond to the business cycle is clearly sub-optimal, just like a time-invariant emission tax. However, such a cap would also fix output and the interest rate over the business cycle, which renders this model not suitable for an analysis of optimal monetary policy. To complicate matters even further, the EU emission trading system allows firms to "bank" on emission permits. Firms can acquire permits when they are cheap, for example during a recession, and use them when they are more expensive during a boom.[10] On one hand, this would imply pro-cyclical emissions also under an emission trading system, consistent with the empirical observations in Fig. 1. Thus, we argue that our qualitative results are at least broadly applicable to quantity-based regulation as well. On the other hand, a much richer setting with precautionary permit banking and technology choice is required to analytically flesh out all relevant transmission channels of monetary policy and the different components of the central bank loss function.

Related literature Our paper draws from the environmental DSGE literature that studies the macroeconomic effects of climate policies at business cycle frequencies, starting with the contribution by Heutel (2012). Faria et al. (2022) discuss the environmental neutrality of monetary policy under different monetary frictions, such as cash-in-advance or money-in-the-utility function. We contribute to a growing literature studying how monetary policy optimally adapts to climate change. McKibbin et al. (2020) provide an overview about potential interactions between climate policy and monetary policy. For a general discussion of these interactions, we also refer to Hansen (2021).

Within the E-DSGE literature, our paper is closely related to the environmental New Keynesian literature, which studies the relationship between environmental and monetary policies, see Annicchiarico et al. (2021) for a survey. Annicchiarico and Di Dio (2017) use a calibrated environmental New Keynesian model with endogenous emission abatement to study Ramsey-optimal monetary and environmental policy. They show that perfect price stabilization is generally not optimal, which is consistent with the main analytical findings in our setup. Muller (2023) proposes an adjustment to the natural interest rate for environmental damages, measured by air pollution and carbon dioxide emissions and shows that pro-cyclical emissions are an important driver of the difference between this adjusted "green interest rate" and the natural interest rate. By tracking the "green interest rate", monetary policy can intertemporally re-allocate consumption from periods with high pollution intensity to periods with a low pollution intensity. Nakov and Thomas (2023) show that climate change has a quantitatively limited impact on conventional monetary policy, even if emission taxes are sub-optimally low. Economides and Xepapadeas (2025) study monetary policy numerically in a larger E-DSGE model, where positive TFP shocks have negative side effects through elevated damages from climate change. Our paper differs from these papers by focusing on optimal monetary policy through an analytical characterization of central bank loss functions.

A growing literature studies monetary policy when inflation is partially driven by rising energy prices. In a New Keynesian model with an energy sector, Olovsson and Vestin (2023) show that targeting core inflation is welfare-maximizing. The literature also recognizes that monetary policy might be affected by potentially inflationary effects of carbon taxation more generally. Konradt and Weder di Mauro (2023) and Hensel et al. (2024) provide empirical evidence. Ferrari and Nispi Landi (2022), Del Negro et al. (2023) and Airaudo et al. (2024) study this channel through the lenses of small- to medium-scale New Keynesian models, while Sahuc et al. (2024) estimate a New Keynesian E-DSGE model to assess the macroeconomic relevance of "climateflation" and "greenflation".

Outline Our paper is structured as follows. Section 2 presents a four-equation New Keynesian model, augmented by a law of motion for socially harmful emissions. Section 3 demonstrates that the natural and welfare-relevant output gaps can be reasonably approximated in a reduced three-equation version of the model and then characterizes the competitive equilibrium. In Section 4, we derive the central bank loss function and study optimal monetary policy under discretion in closed form. Section 5 provides quantitative results and comparative statics with respect to macro and climate parameters. Section 6 concludes.

[9] We also refer to Benmir et al. (2023) for a richer quantitative DSGE model of cyclical emission permits and excessive permit price volatility.

[10] For a comprehensive discussion of permit banking, we refer to Kollenberg and Taschini (2019) and the references therein. Fuchs (2025) provides empirical evidence that precautionary holdings of emission permits are widespread in the EU.

2. Model

We present the basic monetary policy trade-off in an otherwise standard New Keynesian model, augmented by socially harmful emissions. There is a representative household, monopolistically competitive firms, a fiscal authority, and the central bank. Emissions negatively affect the productivity of final good producers through a damage function.[11]

Households The representative household saves using nominal deposits S_t that pay the one-period gross interest rate r_t^s, consumes the final consumption good c_t, and supplies labor n_t at the nominal wage W_t. The household also owns firms and receives their profits d_t^{firms}, expressed in real terms. The maximization problem is given by

$$\max_{\{c_t,n_t,S_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}-1}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right) \right]$$
$$\text{s.t.} \quad P_t c_t + S_t = W_t n_t + r_{t-1}^s S_{t-1} + P_t d_t^{firms}.$$

The parameters σ and φ determine the inverse of, respectively, the intertemporal elasticity of substitution and the elasticity of labor supply. Solving this maximization problem yields a standard Euler equation and an intra-temporal labor supply condition

$$c_t^{-\sigma} = \beta r_t^s \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{\pi_{t+1}} \right] , \tag{1}$$

$$n_t^{\varphi} = w_t c_t^{-\sigma} . \tag{2}$$

Here, P_t is the price level, $w_t \equiv \frac{W_t}{P_t}$ is the real wage, and $\pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes gross inflation.

Firms There is a mass-one continuum of monopolistic firms, indexed by i. Firm i hires labor $n_t(i)$ to produce the intermediate good $y_t(i)$ with the following technology:

$$y_t(i) = \Lambda_t A_t n_t(i) . \tag{3}$$

Here A_t is exogenous total factor productivity which follows an AR(1) process in logs:

$$\log(A_t) = \rho_A \log(A_{t-1}) + \sigma_A \epsilon_t^A , \quad \text{where } \epsilon_t^A \sim N(0,1).$$

Production revenues are taxed at the rate τ_t^c, which we will link to socially harmful emissions during the production process, described below. Production damages associated with emissions are summarized by the term Λ_t, which we also describe below. Firms are not always able to adjust their prices due to nominal rigidities, modeled as in Calvo (1983), with θ being the fraction of firms that is not allowed to change prices. The optimal price for a firm that is able to adjust prices is given by

$$p_t^* = \frac{1}{1-\tau_t^c} \mu \frac{\xi_{1,t}}{\xi_{2,t}} \tag{4}$$

where

$$\mu = \frac{\epsilon}{\epsilon-1}, \quad \xi_{1,t} = mc_t\, y_t + \beta\, \theta\, \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon} \xi_{1,t+1} \right], \quad \text{and} \quad \xi_{2,t} = y_t + \beta\, \theta\, \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon-1} \xi_{2,t+1} \right].$$

Carbon taxes enter the model by increasing firms marginal cost, which implies that the optimal price of an adjusting firm p_t^* increases in the carbon tax τ_t^c. The nominal friction implies that monopolistic producers face time-varying real marginal costs, thus generating a relationship between inflation and real economic activity summarized in the New Keynesian Phillips Curve.[12]

Emission damages During the production process, firms emit one unit of socially harmful emissions per unit of output. The stock of atmospheric carbon dioxide E_t evolves according to

$$E_t = y_t + (1-\delta) E_{t-1} , \tag{5}$$

where $\delta > 0$ is the depreciation rate of atmospheric carbon dioxide. Following the literature, the stock of emissions endogenously reduces productivity:

$$\Lambda_t = \exp\left\{ -\Gamma E_t \right\} , \tag{6}$$

where the parameter Γ governs the severity of damages associated with emissions. Importantly, damages are an externality, because they depend on aggregate economic activity y_t, which individual firms take as given. Our analysis abstracts from technological change or abatement effort at the firm level. The full list of equilibrium conditions is summarized in Appendix A.1, Eqs. (A.47)–(A.57).

[11] Analytically similar results can be obtained by assuming that emissions exert a utility loss on households. In this case, the competitive equilibrium under flexible prices also overreacts to TFP shocks.

[12] Similar results can be obtained by imposing price adjustment costs instead of staggered pricing.

Table 1
Parameterization.

Parameter	Value	Source/Target
Households		
Household discount factor β	0.995	Real rate 1% p.a.
Consumption CRRA σ	1	Log-utility
Labor supply curvature φ	4/3	Chetty et al. (2011)
Demand elasticity final goods ϵ	6	20% markup
Technology		
Calvo parameter θ	0.8	Price Duration 5 Quarters
Taylor-rule parameter ϕ	1.5	Standard
Emission damage γ	0.37	Bilal and Kaenzig (2024)
Emission decay δ	0.0035	Gibson and Heutel (2023)
Shocks		
Persistence TFP ρ_A	0.9	Standard
TFP shock st. dev. σ_A	0.01	Standard

Before we turn to the model parameterization, some remarks on emission damages are in order. Our quantitative application focuses on the negative effects of carbon emissions through climate change, i.e. economic damages depend on the stock of emissions. Our theoretical analysis of externalities in the New Keynesian model is also applicable to other damages associated with harmful emissions beyond their contribution to climate change. The environmental economics literature typically views climate change as only a subset of the overall adverse effects that the emissions exert on the wider economy. Other adverse effects include negative health consequences through air quality losses, decreased timber and agriculture yields, depreciation of materials, and reductions of recreation services. For details, we refer to Muller et al. (2011) and the references therein. In contrast to climate change, these negative effects materialize very quickly in response to a cyclical increase in economic activity, but also depreciate faster, such that it might be more appropriate to assume that damages depend on the flow of emissions.

These alternative sources of emission damages have different quantitative implications for macroeconomic dynamics. In particular, the depreciation rate of pollutants affects how quickly damages increase after an expansionary productivity shock. This in turn governs the output response to a shock and also the *natural rate* of interest consistent with flexible prices. However, what matters for optimal monetary policy is the welfare-relevant output gap and not the output response per se. In the next section, we will show that emission damages drive a very similar wedge between efficient and actual output, irrespective of how fast emissions translate into *current* damages. This is because the planner takes the *present value* of all future damages into account when deciding on the socially optimal output expansion. If pollution damages depend on a slowly depreciating stock of emissions, output expansions have only little effect on current damages, but they have a long-lasting effect on damages. Conversely, if damages depend on the flow, there is large negative effect on impact but only a negligible effect in future periods. Quantitatively, the present value of damages turns out to be quite similar in either case, as we show in the next Section.

Market clearing, monetary policy and welfare Lastly, market clearing requires $c_t = y_t$ since we abstract from physical capital and investment. The model is closed by an additional equation that links inflation to the nominal interest rate. As a reference point to discuss the model's positive properties, we use Taylor-type rule:

$$r_t^s = \bar{r}^s \cdot \pi_t^{\phi} \,. \tag{7}$$

where ϕ governs the response of the short term nominal interest rates to inflation and $\bar{r}^s = 1/\beta$ is the steady state real interest rate. Once we study optimal monetary policy, the monetary policy rule is derived from first principles, i.e. it maximizes household welfare, which is defined recursively through:

$$V_t = \log(c_t) - \frac{n_t^{1+\varphi}}{1+\varphi} + \beta \mathbb{E}_t[V_{t+1}] \,. \tag{8}$$

Parameterization While all of our main results are obtained analytically, we illustrate their quantitative relevance under a standard quarterly parameterization following the New Keynesian DSGE literature. Households' risk aversion and discount factor are set to $\sigma = 1$ and $\beta = 0.995$. This discount factor implies an annual real rate of 2%. Furthermore, we set the curvature in labor supply disutility to $\varphi = 4/3$, following Chetty et al. (2011).

Regarding the emission externality, we use a narrow interpretation as carbon emissions and target recent estimates by Bilal and Kaenzig (2024), who identify a long run productivity loss of 31% under the current trajectory of carbon dioxide emissions. We perform a change of variables and define a damage parameter $\gamma \equiv \frac{\Gamma}{E}$ adjusted for steady state emissions. Then, setting $\gamma = 0.37$ yields a long run value of $\Lambda = \exp(-0.37) = 0.69$ for the damage function. Following Gibson and Heutel (2023), we set the quarterly depreciation rate of the emission stock to $\delta = 0.0035$, which implies a 50 year half-life for atmospheric carbon dioxide.

The demand elasticity for final good varieties is fixed at $\epsilon = 6$, implying a 20% markup. As a benchmark, we set the Calvo parameter to $\theta = 0.8$, corresponding to an expected price duration of five quarters. In a comparative statics exercise, we decrease this parameter to $\theta = 0.6$, implying an expected price duration of 2.5 quarters. The parameter in Eq. (7) governing the interest rate response to inflation is set to ϕ=1.5. Lastly, the parameters governing exogenous TFP are set to $\rho_A = 0.9$ and $\sigma_A = 0.01$. The parameterization is summarized in Table 1.

3. Exogenous monetary policy

We first discuss the positive properties of augmenting an otherwise standard New Keynesian model with socially harmful emissions by keeping the monetary policy reaction function constant, that is we close the model by a Taylor-type rule. To focus on the emission externality we assume for the remainder of this paper that the fiscal authority sets a constant labor subsidy $\tau^n = \frac{1}{\epsilon}$, which is financed by lump-sum taxes and implies that $(1-\tau^n)\mu = 1$, which eliminates the steady state distortion generated by monopolistic competition. Since emission externalities are the only inefficiency in the flexible price equilibrium, we can then cleanly distinguish between natural and efficient (welfare-relevant) output gap.

3.1. Efficient and natural output gap

We begin by characterizing efficient output y_t^e and natural output y_t^n, defined as the output consistent with perfectly flexible prices.

Proposition 1. *The natural y_t^n and efficient y_t^e output levels can be written as:*

$$(y_t^n)^{\sigma+\varphi} = (1-\tau_t^c)(A_t\Lambda_t)^{1+\varphi} \ . \tag{9}$$

$$(y_t^e)^{-\sigma} - \frac{(y_t^e)^{\varphi}}{(A_t\Lambda_t)^{1+\varphi}}\left(1+\gamma\frac{y_t^e}{E}\right) = \beta(1-\delta)\left((y_{t+1}^e)^{-\sigma} - \frac{(y_{t+1}^e)^{\varphi}}{(A_{t+1}\Lambda_{t+1})^{1+\varphi}}\right) . \tag{10}$$

Proof. see Appendix A.1.

These output responses serve as reference in the definition of the efficient $x_t^e \equiv \hat{y}_t - \hat{y}_t^e$ and the natural output gap $x_t^n \equiv \hat{y}_t - \hat{y}_t^n$, respectively. Here $\hat{y}_t$ is the log deviation of output from its steady state level for any given degree of nominal rigidities and any given monetary policy. Without externalities ($\gamma = 0$), the flexible price competitive equilibrium coincides with the efficient allocation, such that both output gaps coincide $x_t^e = x_t^n$.

With externalities ($\gamma > 0$), an overreaction of the flexible price economy in response to a positive TFP shock implies a positive efficient output gap, illustrated in the left panels of Fig. 2. The solid red line in the top left panel corresponds to the output expansion in the flexible price equilibrium, which exceeds the efficient output expansion, indicated by the solid green line. The output expansion with sticky prices is smaller than in the flexible price case, as shown by the dashed blue line for a Calvo parameter of $\theta = 0.6$ and the dotted blue line for $\theta = 0.8$. Consequently, as the middle left panel of Fig. 2 shows, the natural output gap in response to a positive TFP shock is negative.

The bottom left panel shows that it jointly depends on the extent of nominal rigidities and the emission externality whether the competitive equilibrium still overreacts relative to the efficient allocation. For a large θ, nominal rigidities dominate and the efficient output gap is also negative, albeit smaller. For smaller values of θ, the overreaction of output with respect to the efficient allocation dominates and the efficient output gap turns positive. In our parameterization, this happens for a Calvo parameter between 0.6 and 0.8, i.e. for empirically relevant parts of the parameter space. It will turn out that the interaction of these two dynamic inefficiencies, nominal rigidities and emission externalities, is non-trivial and bears direct implications for the conduct of monetary policy.

3.2. Eliminating the fourth equation

An analytical characterization of this model is complicated by the law of motion for emissions. While the persistence of socially harmful emissions has important implications for macroeconomic dynamics, it turns out that it is less relevant for the welfare-relevant output gap, which is crucial for monetary policy. Specifically, we demonstrate that the dynamics of both output gaps can be accurately approximated by assuming that damages depend on the flow of emissions, i.e. current output, rather than the stock of cumulated emissions. This will allow us to characterize optimal monetary policy and the central bank loss function in closed form in the next section.

The right panels of Fig. 2 reveal that eliminating the law of motion for emissions by setting $\delta_E = 1$ delivers a reasonably good approximation of the efficient and natural output gap. In this special case, damages depend on a flow pollutant, i.e. we have $E_t = y_t$ and the model reduces to three equations. Notably, the output reaction $\hat{y}_t$ is smaller in the model with full depreciation, as damages Λ_t are more responsive to current economic activity. However, the response of actual and efficient output relative to the flexible price output reaction are remarkably similar. The reason is that the social planner takes the *present value* of future damages into account when computing the optimal output expansion. This is also in line with the concept of the social cost of carbon, which collects the present value of all economic damages implied by a pulse of socially harmful emissions.

In the following, we analytically characterize the competitive equilibrium of the three-equation model with a flow pollutant.

Lemma 1. *If $\delta_E = 1$, the efficient output level y_t^e reduces to:*

$$(y_t^e)^{\sigma+\varphi} = \frac{(A_t\Lambda_t)^{1+\varphi}}{1+\gamma\frac{y_t^e}{y}} \ . \tag{11}$$

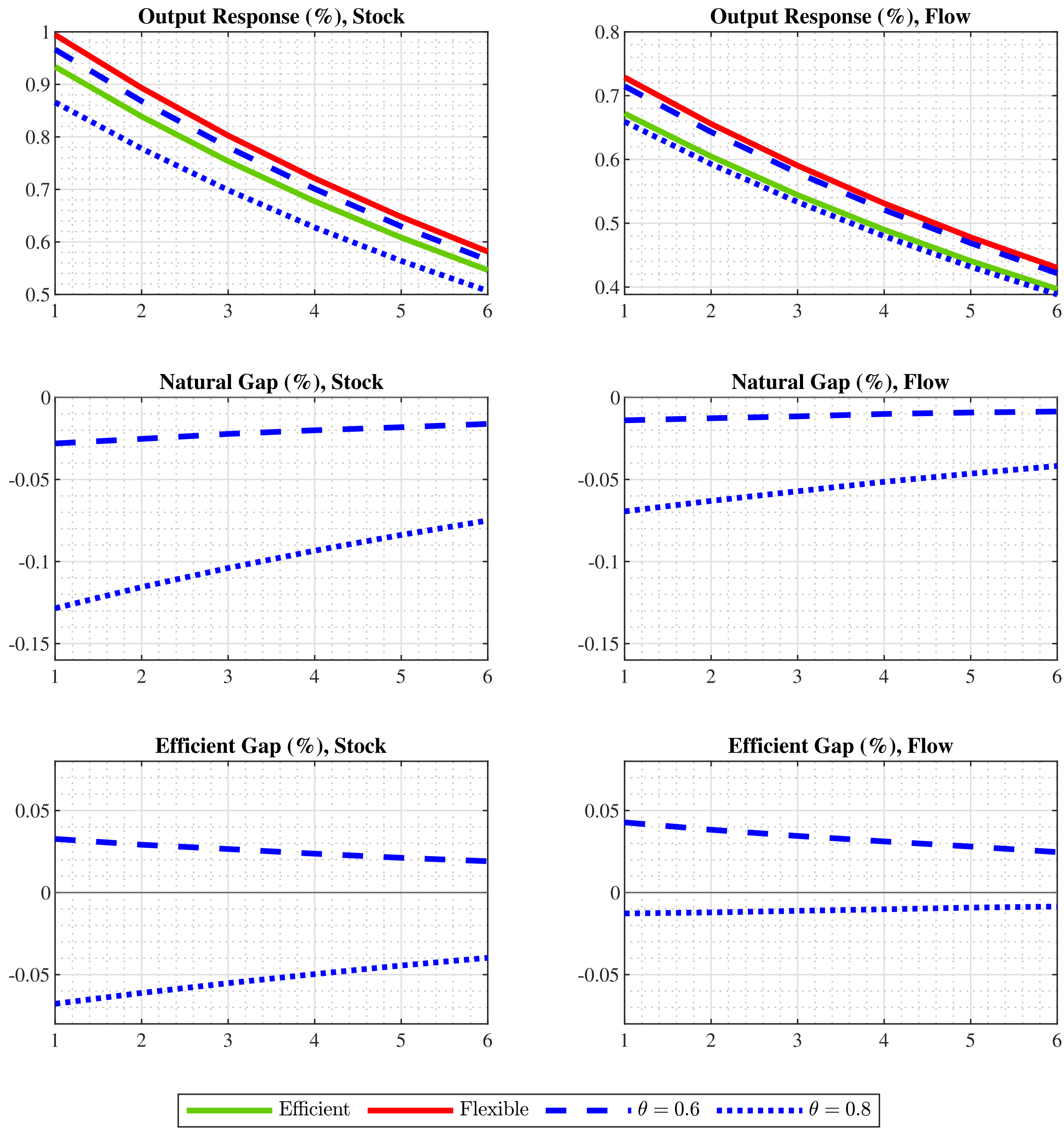

Fig. 2. IRF to TFP-Shocks in the Three- and Four-Equation Model. Impulse responses to a positive one standard deviation shock to TFP over 6 quarters. The left panel refers to the four-equation model with a stock pollutant. The right panel refers to the three-equation model with a flow pollutant. The output change $\hat{y}_t$ is expressed in relative deviations from its steady state value. The natural output gap is defined as the difference between the actual output reaction (dotted or dashed blue lines, respectively) and the flexible price allocation (red line in top row). The efficient output gap is defined with respect to the efficient allocation (green line in top row). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

This immediately obtains from simplifying (10) and implies that we can write both efficient and natural output as a function of the only remaining state variable A_t. Combining (11) with (9), the ratio of natural and efficient output simplifies to

$$\left(\frac{y_t^n}{y_t^e}\right)^{\sigma+\varphi} = \left(1 + \gamma \frac{y_t^e}{y}\right)(1 - \tau_t^c) \,. \tag{12}$$

The log-deviations of $\hat{y}_t^n$ and $\hat{y}_t^e$ around the deterministic steady state are given by:

$$\hat{y}_t^n = \frac{1+\varphi}{\zeta} a_t - \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c \tag{13}$$

$$\hat{y}_t^e = \frac{1+\varphi}{\tilde{\zeta}+\tilde{\gamma}} a_t \ , \tag{14}$$

where

$$\zeta \equiv \varphi + \gamma(1+\varphi) + \sigma, \quad \tilde{\zeta} \equiv \varphi + \gamma\frac{y^e}{y}(1+\varphi) + \sigma \quad \textit{and} \quad \tilde{\gamma} \equiv \frac{\gamma}{1+\gamma} \ . \tag{15}$$

Proof. see Appendix A.2.

It follows from Eq. (12) that a time-varying emission tax $\tau_t^c = \frac{\gamma \frac{y_t^e}{y}}{1+\gamma\frac{y_t^e}{y}}$ implements the efficient allocation. In this case, the RHS of (12) collapses to one and natural and efficient output coincide. Optimal emission taxes are pro-cyclical, which resembles the result of Golosov et al. (2014), who show that optimal emission taxes are proportional to GDP growth. The optimality of pro-cyclical emission taxes also arises in the workhorse environmental RBC model (Heutel, 2012).

Absent emission taxes ($\tau_t^c = 0$), the natural level of output generally exceeds its efficient level. However, even with a emission tax implementing the efficient steady state output, emissions still generate a dynamic inefficiency whenever emission taxes do not respond appropriately to the business cycle.[13] Specifically, with $\tau^c = \tilde{\gamma}$ and $\hat{\tau}_t^c = 0$, output in the competitive equilibrium $\hat{y}_t^n$ overreacts to technology shocks relative to the efficient allocation $\hat{y}_t^e$. In this case, we have $\tilde{\zeta} = \zeta$, implying $\hat{y}_t^e = \frac{1+\varphi}{\zeta+\tilde{\gamma}} < \frac{1+\varphi}{\zeta} = \hat{y}_t^n$. Since this dynamic inefficiency is the key element of our analysis, we focus on the empirically plausible case of constant carbon taxes ($\hat{\tau}^c = 0$) in the following characterization of monetary policy. Furthermore, Lemma 1 demonstrates that, all else equal, a positive carbon tax shock is recessionary. The recessionary effect of the carbon tax precisely addresses over-production in the competitive equilibrium.

3.3. Characterization of the three-equation model

As a next step, we flesh out the interactions between nominal rigidities and pro-cyclical emissions for an exogenously given nominal interest rate r_t^s in the three-equation model. Proposition 2 shows that its equilibrium is characterized by a dynamic IS curve and the New Keynesian Phillips curve (NKPC).

Proposition 2. *The equilibrium conditions for the economy with nominal rigidities simplify to the following two linear conditions in terms of log-deviations from the steady-state:*

$$x_t^n = \mathbb{E}_t[x_{t+1}^n] - \frac{r_t^s - \mathbb{E}_t[\pi_{t+1}]}{\sigma} + \underbrace{\frac{1}{\zeta}\mathbb{E}_t\left[(1+\varphi)(a_{t+1}-a_t) - \frac{\tau^c(\hat{\tau}_{t+1}^c - \hat{\tau}_t^c)}{1-\tau^c}\right]}_{\equiv r_t^n/\sigma} , \tag{16}$$

$$\pi_t = \zeta\kappa x_t^n + \beta\mathbb{E}_t[\pi_{t+1}] + \beta(1-\theta)\frac{\tau^c}{1-\tau^c}(\hat{\tau}_t^c - \mathbb{E}[\hat{\tau}_{t+1}^c]) \ . \tag{17}$$

Proof. see Appendix A.3.

Eq. (16) is a dynamic IS curve: the (natural) output gap x_t^n positively depends on the expected output gap next period and negatively depends on the real interest rate gap $\frac{r_t^s - \mathbb{E}_t[\pi_{t+1}] - r_t^n}{\sigma}$, defined as the real interest rate, $r_t^s - \mathbb{E}_t[\pi_{t+1}]$, minus the natural real interest rate, r_t^n. The natural interest rate is the real interest rate consistent with the natural level of output.[14] The externality negatively affects the transmission of a productivity shock on the natural output gap and, hence, the natural interest rate. Intuitively, the externality dampens the response to productivity shocks, as productivity endogenously declines, counteracting the initial effect of te shock. However, it turns out that the quantitatively relevant and economically interesting implications of real externalities operate through the NKPC and the central bank loss function.

The NKPC is given by Eq. (17) and relates the natural output gap to inflation. As usual, its slope depends on preference parameters and nominal rigidities through $\zeta\kappa = \zeta\frac{(1-\theta\beta)(1-\theta)}{\theta}$. In the standard model, the auxiliary parameter ζ reduces to the familiar expression $(\sigma+\varphi)$. In the presence of emission externalities, the auxiliary parameter ζ is also affected by the externality parameter γ since $\zeta = \sigma + \varphi + \gamma(1+\varphi)$. Consequently, pro-cyclical emissions steepen the NKPC. Intuitively, the inflation response to a TFP shock is

[13] Note that our analysis is based on a stationary model. As customary in the (New Keynesian) business cycle literature, the data counterparts of our model variables are trend deviations and the concept of pro-cyclical emissions in Fig. 1 is based on trend deviations of GDP and emissions. If climate policy is instead modeled in terms of a transition towards higher emission taxes, optimal taxes should still be above (below) their trend during a boom (recession). As long as the carbon taxes do not deviate from their trend in response to business cycle fluctuations, the dynamic inefficiency arises.

[14] As a by-product of our analysis, Eqs. (16) and (17) characterize the effects of carbon taxes on inflation and output. Specifically, transitory carbon tax shocks, i.e. $\hat{\tau}_t^c > 0$ and $\hat{\tau}_{t+1}^c = 0$, are both inflationary and recessionary. This result is consistent with empirical findings in Kaenzig (2023) and is related to the negative effect of carbon taxes on marginal costs, which implies that, on aggregate, firms reduce their production and increase their prices.

determined by the share of firms that can reduce their price, which does not depend on the emission externality. However, the emission externality dampens the effects of a TFP shock on the natural output gap, as the externality-adjusted productivity $\Lambda_t A_t$ is smaller than in the standard model without the externality. This implies that, fixing the inflation response, the natural output gap is less responsive to a productivity shock. Vice versa, for a given natural output gap, inflation responds more strongly to a TFP shock if $\gamma > 0$.

Note that these inflationary pressures, sometimes referred to as *climateflation*, do not imply that the emission externality is inflationary in equilibrium, as firms take expected inflation in future periods into account when setting their prices today, which in turn depends on the central bank's reaction to shocks. It is, therefore, necessary to iterate forward the NKPC when characterizing the equilibrium effect of TFP shocks. Proposition 3 characterizes how pro-cyclical emissions affect output and inflation in the competitive equilibrium for the constant central bank reaction function (7).

Proposition 3. *Under time-invariant emission taxes, the policy functions for output gap and inflation read*

$$x_t^n = \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{xa} a_t \,, \tag{18}$$

$$x_t^e = \widetilde{\gamma}\frac{1+\varphi}{\zeta(\zeta+\widetilde{\gamma})} + \Theta_{xa} a_t \,, \tag{19}$$

$$\pi_t = \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{\pi a} a_t \,. \tag{20}$$

Moreover, the variances of output gap and inflation are given by:

$$Var[x_t^n] = \Theta_{xa}^2 \sigma_A^2, \quad Var[\pi_t] = \Theta_{\pi a}^2 \sigma_A^2.$$

Proof. See Appendix A.4.

Fig. 3 illustrates Proposition 3 graphically. The first row shows the impact response of inflation and output gap to a positive technology shock as a function of θ. The standard case without emission externalities is indicated by the dashed yellow line. The output gap (inflation) is larger (smaller) in absolute terms as θ increases, i.e. as prices become more rigid. For the case with emission externalities we differentiate between the natural output gap x_t^n (red line) and the efficient output gap x_t^e (green line), which coincide for the standard model. Due to the dampening effect of emissions on TFP, the inflation and natural output gap are less responsive to productivity shocks in the model with emissions but always negative. The response of the efficient output gap (19) is similar to the natural output gap, but shifted by a positive intercept term. The efficient output gap turns positive for $\theta < 0.78$, consistent with Fig. 2.

In the second row, we plot the responses of both output gaps and inflation for different values for the emission externality. Differentiating the coefficients of the policy functions (18) and (20) with respect to γ, we immediately obtain $\frac{\partial \Theta_{xa}}{\partial \gamma} > 0$ and $\frac{\partial \Theta_{\pi a}}{\partial \gamma} > 0$. The response of the natural output gap (18) and inflation (20) to a TFP shock are *smaller in absolute terms* if the externality is more severe, see the red lines in the lower panel of Fig. 3. This decline in macroeconomic volatility is associated with the dampening effect that the emission externality exerts on the aggregate production function. The efficient output gap is increasing in γ, since both the natural output gap and the intercept in Eq. (19) increase in γ. The intercept term in (19) is reflected by the difference between the green and red line. In the simplified three-equation model, the efficient output gap turns positive for $\gamma > 0.08$.

4. Optimal monetary policy: LQ-approach

Having discussed how pro-cyclical emissions affect the competitive equilibrium allocation and the efficient allocation, we now analyze optimal monetary policy. We derive the central bank objective analytically by extending the methodology outlined in Benigno and Woodford (2005) for the case of inefficient competitive equilibria under flexible prices. Since the loss function is available in closed form and easily interpretable, we can take a linear-quadratic approach to analytically show the optimal response of monetary policy to i.i.d. productivity shocks. Lastly, we consider the case of demand shocks and exogenous cost push shocks.

4.1. Central bank loss function

The central bank objective function is derived from first principles, i.e. we maximize a utilitarian welfare maximization problem which is closely linked to the distinction between the efficient and natural output gap described in Proposition 1. Since over-production in the competitive equilibrium allocation is at the heart of the mechanism, we follow Benigno and Woodford (2005) and consider the general case where the steady-state level of output and labor are above their efficient levels.

Proposition 4. *A second order approximation of the welfare function around the distorted steady state yields the following quadratic loss function:*

$$\mathcal{L} = -\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \frac{U_t - U}{U_c C}\right] \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \left\{ \pi_t^2 + \omega_x \left(x_t^n - x_t^*\right)^2 \right\}\right] + t.i.p. \tag{21}$$

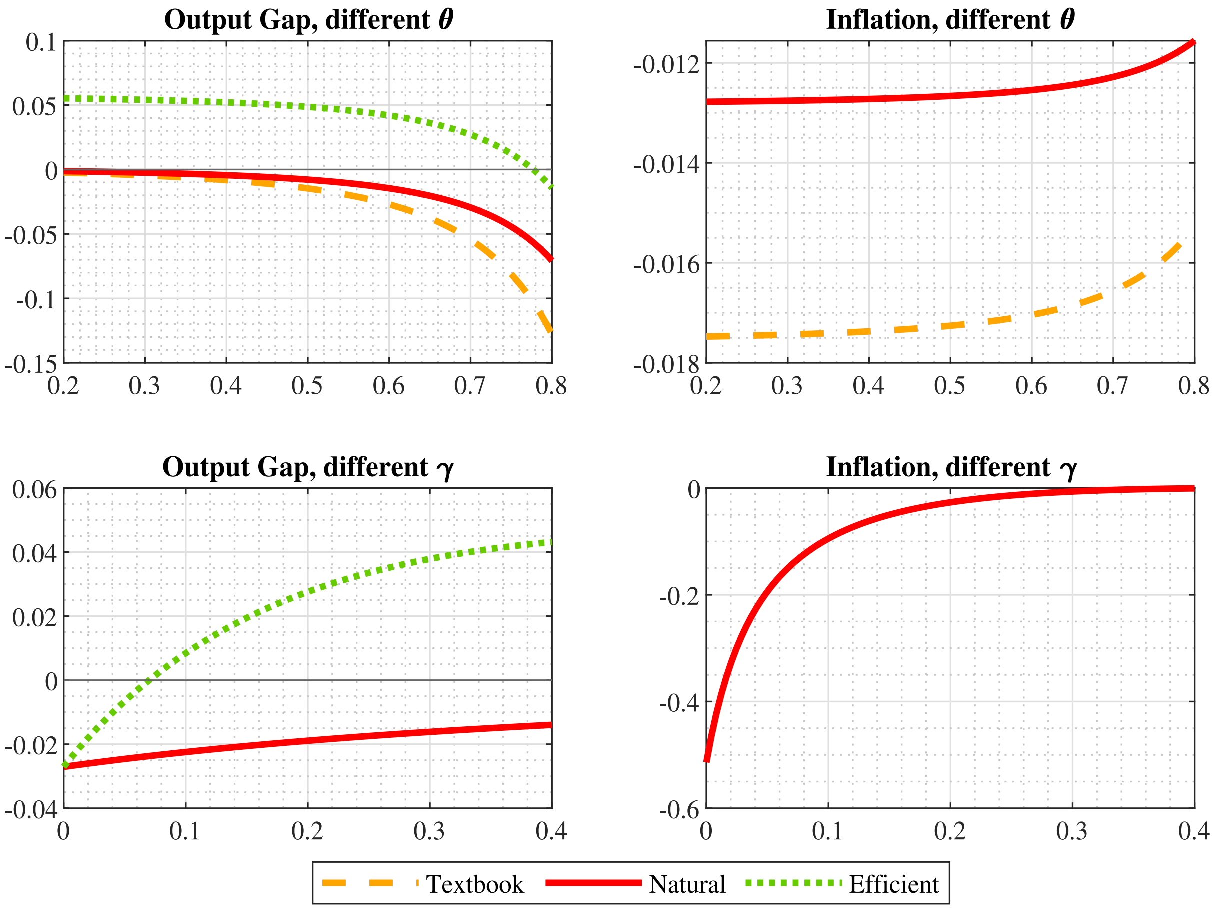

Fig. 3. Inflation and Output Gap Responses as Functions of θ and γ. The upper panel displays the response to a one standard deviation TFP shock for different Calvo parameters θ. In the upper left, we show the natural output gap (Eq. (18), solid red line) and the efficient output gap (Eq. (19), dotted green line) with the externality set to $\gamma = 0.37$. The dashed orange line refers to the standard model without the externality where natural and efficient output gap coincide. The upper right shows the response of inflation (Eq. (20)) for the model with the externality (solid red line) and the standard model (dashed orange line). In the lower panel, we show the responses to a one standard deviation TFP shock for different values of the damage parameter γ, fixing the Calvo parameter at $\theta = 0.8$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where x_t^* is the target level of the natural output gap and ω_x is the weight on output stabilization:

$$\omega_x = \frac{\kappa}{\epsilon}(\zeta(1+\gamma)+\gamma)\,, \tag{22}$$

$$x_t^* = \frac{\Omega_{xa}}{\Omega_x}a_t = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma}a_t\,. \tag{23}$$

Proof. see Appendix A.5.

The proof is an extension of Benigno and Woodford (2005) to the case of inefficient competitive equilibria under flexible prices. The planner solution takes this inefficiency into account by internalizing the marginal effect of economic activity on emission damages. It is essential that this enters the second order approximation of the household utility function as an additional variable. If the central bank took the relationship $\Lambda_t = \exp(-\gamma y_t)$ as given, the resulting loss function would prescribe to close inflation and natural output gap at all times. Put differently, monetary policy would (sub-optimally) go along with the market failure in the flexible price allocation and the loss function would inherit this inefficiency. We relegate the analytical steps to Appendix C. In Section 5, we quantitatively evaluate the welfare cost of ignoring the pro-cyclicality of emissions in the conduct of monetary policy.

Once the additional term accounting for the marginal effect of economic activity on emission damages enters the loss function, it introduces several linear terms that have to be taken care of appropriately. In addition to a second order approximation of the NKPC that is already necessary due to the potentially distorted steady state, we show that it is also necessary to take a second order approximation of the relationship between economic activity and emission damages. In this case, it is possible to derive a

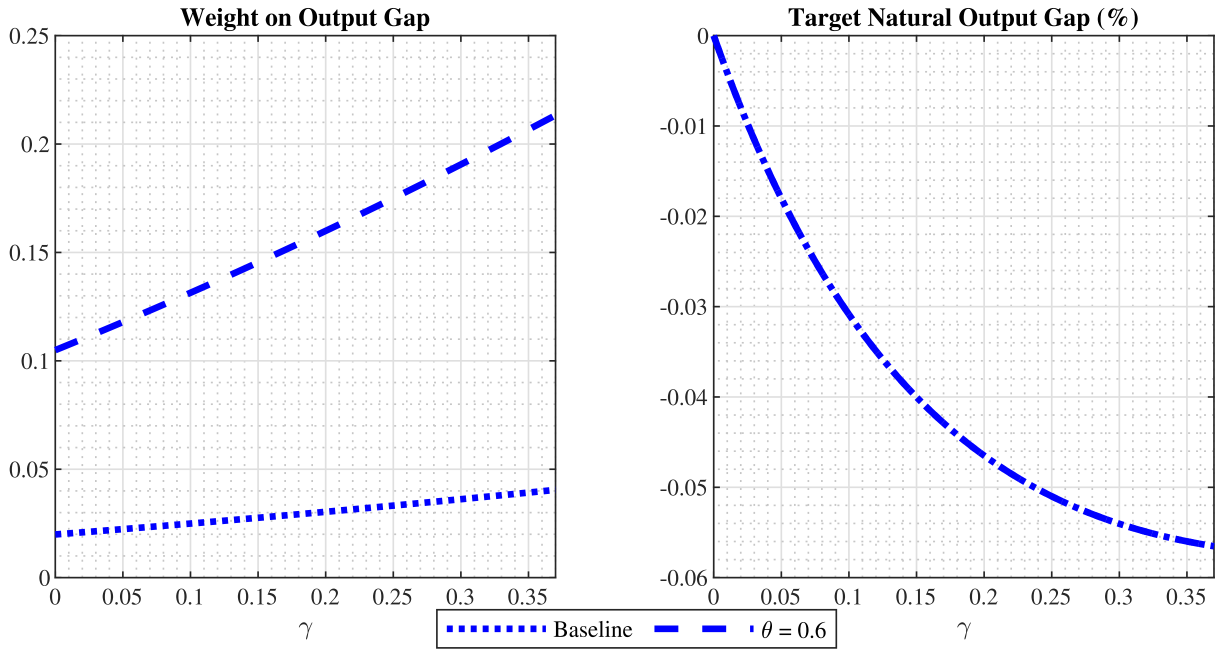

Fig. 4. Weight and Target Natural Output Gap as Functions of γ. Target level of the natural output gap x_t^* (right panel) and weight ω_x on the natural output gap x_t^n (left panel) in the central bank loss function (21) for different externality parameters γ. The dotted line corresponds to the baseline Calvo parameter of $\theta = 0.8$, while the dashed line reflects the case of $\theta = 0.6$.

loss function (21) that completely separates business cycle stabilization objectives from distortions in the steady state, analogously to Benigno and Woodford (2005). As in their analysis, the steady state wedge does not affect the conduct of macroeconomic stabilization policy, which is solely concerned with addressing the dynamic inefficiencies associated with pro-cyclical emissions and price rigidities.

To gain intuition behind Proposition 4, it is helpful to first consider the standard case absent the emission externality. In this case, the target level of the output gap x_t^* collapses to zero: efficient and natural output gap coincide and it is optimal to close the output gap at all times. Furthermore, the weight on the output gap ω_x in the loss function reduces to the familiar expression

$$\omega_x = \frac{\kappa}{\epsilon}(\sigma + \varphi)\,,$$

where the auxiliary parameter $\kappa = \frac{(1-\theta\beta)(1-\theta)}{\theta}$ decreases in the share θ of firms that cannot adjust prices. A high κ reflects mild nominal rigidities and comparatively large inflation responses to shocks, such that the weight on output stabilization increases in κ. Graphically, the dashed line in the left panel of Fig. 4, reflecting milder nominal rigidities with $\theta = 0.6$, lies above the dotted line, reflecting more severe nominal rigidities with $\theta = 0.8$. This holds for every externality parameter, including the special case of $\gamma = 0$. Furthermore, the target level for the natural output gap is zero in this case, see the right panel of Fig. 4.

In the model with an emission externality, the central bank places a higher weight on output stabilization if the externality is more severe. This is indicated by the left panel of Fig. 4 and immediately follows from Eq. (22) for any degree of price rigidities. Intuitively, the degree of over-production in competitive equilibrium is positively related to severity of the emission externality γ while the severity of nominal rigidities is unaffected by the externality, such that output gap stabilization becomes more relevant from a welfare perspective if γ is large.

The right panel of Fig. 4 illustrates that the target level of the natural output gap is negatively related to the externality parameter γ but independent of the Calvo parameter, which does not enter Eq. (23). Again, a more severe externality renders a larger deviation from a zero natural output gap optimal. Importantly, Proposition 4 also shows that the target level for the natural output gap x_t^* in response to an expansionary productivity shock is *negative*. This finding is conceptually related to the inflationary bias in monetary policy theory, which is discussed for example in Clarida et al. (1999) and stems from the desire to push output above the natural level. By contrast, it is optimal to keep output below the natural level after an expansionary shock in our setting, so that the opposite result emerges. The central bank has a deflationary bias in the presence of real externalities.

4.2. *Optimal monetary policy*

Next, we characterize optimal monetary policy with i.i.d. shocks to TFP. This corresponds to minimizing the loss function derived in Proposition 4 subject to the NKPC, which can be solved for in closed form.

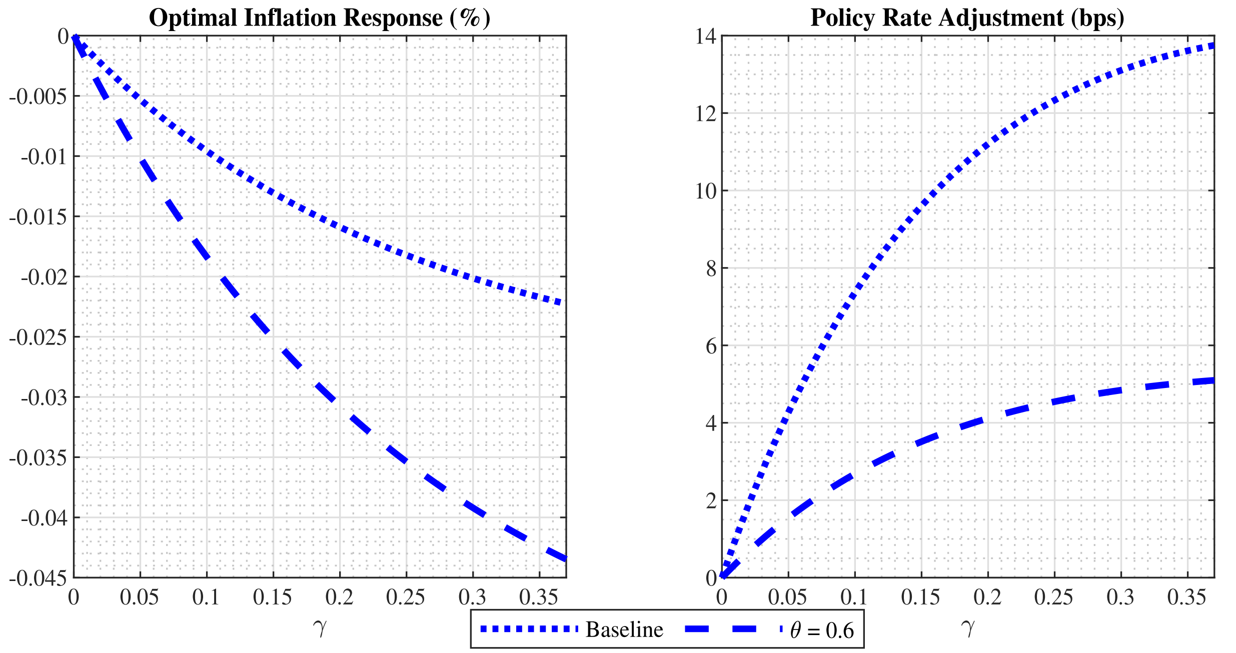

Fig. 5. Optimal Inflation and Interest Rate Adjustment, TFP Shocks. This figure illustrates Proposition 5 by plotting the optimal response of inflation (left panel) and the interest rate adjustment term in Eq. (27) (right panel) to an expansionary markup shock ($\epsilon_t^A = \sigma_A$) as functions of the externality parameter γ. The dotted line corresponds to the baseline Calvo parameter of $\theta = 0.8$, while the dashed line reflects the case of $\theta = 0.6$.

Proposition 5. *If carbon taxes are time-invariant ($\hat{\tau}_t^c = 0$), optimal monetary policy under discretion is characterized by*

$$\pi_t = -\frac{\omega_x}{\zeta\kappa}(x_t^n - x_t^*) \,. \tag{24}$$

Furthermore, if TFP shocks are i.i.d., the optimal responses of inflation π_t^o, natural output gap x_t^o and nominal interest rate r_t^o are given by:

$$\pi_t^o = \frac{\omega_x\kappa\zeta}{\kappa^2\zeta^2+\omega_x}x_t^* = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma}\frac{\omega_x\kappa\zeta}{\kappa^2\zeta^2+\omega_x}a_t \,, \tag{25}$$

$$x_t^o = \frac{\omega_x}{\kappa^2\zeta^2+\omega_x}x_t^* = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma}\frac{\omega_x}{\kappa^2\zeta^2+\omega_x}a_t \,, \tag{26}$$

$$r_t^o = r_t^n - \frac{\sigma\omega_x}{\zeta^2\kappa^2+\omega_x}x_t^* = r_t^n + \frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma}\frac{\sigma\omega_x}{\zeta^2\kappa^2+\omega_x}a_t, \tag{27}$$

where r_t^n is the natural rate of interest implicitly defined through (16).

Proof. see Appendix A.6.

Proposition 5 describes how the central bank optimally resolves the trade-off between overproduction in the flexible price allocation and nominal rigidities. Optimal inflation is negative in response to a positive TFP shock, which can be seen directly from Eq. (25) and is shown graphically in the left panel of Fig. 5. It increases in absolute terms as the externality becomes more severe. Eq. (26) reveals that the optimal (natural) output gap is a linear combination of the target level x_t^* and the flexible price allocation ($x_t^n = 0$).

As shown by Eq. (27), monetary policy does not track the natural interest rate. Instead there is a positive emission adjustment term in Eq. (27) and the central bank implements a smaller interest rate cut in response to a positive TFP shock than in a model without emission externalities. The adjustment term increases in γ, which is graphically illustrated in the right panel of Fig. 5.

Fig. 6 summarizes the effect of emission externalities in a Phillips curve - monetary policy rule diagram. The dashed yellow line refers to the NKPC in the standard New Keynesian model. TFP shocks move the economy on the NKPC, since marginal costs decrease after a positive productivity shocks. Firms find it optimal to decrease prices, generating dis-inflationary pressure. Due to the nominal rigidity, not all firms are able to reduce their prices, so that they face a decreased demand for their goods. After a positive TFP shock, output then increases by less than the natural and efficient level, which coincide in the standard model. Consequently, there is downward pressure on the output gap x_t^n. Since it is optimal (and feasible) to perfectly stabilize inflation and both output gaps, i.e. the dashed black MPR crosses the origin, monetary policy can move the economy back to the origin: divine coincidence holds.

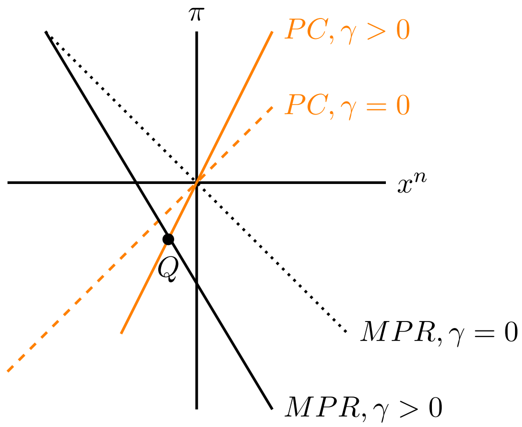

Fig. 6. Phillips Curve - Monetary Policy Rule Diagram. This figure graphically illustrates how the emission externality affects the monetary policy trade-off. The horizontal axis refers to the natural output gap and the vertical axis to inflation. The monetary policy rule is reflected by the dotted black line for the standard case with $\gamma = 0$ and by the solid black line - that does not cross the origin - for the case with emission externalities. The New Keynesian Phillips curve after a positive productivity shock is reflected by the dotted orange line for the standard case with $\gamma = 0$ and by the solid orange line for the case with emission externalities. The equilibrium under optimal monetary policy is characterized by the point Q. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The solid blue line refers to the NKPC in the case with $\gamma > 0$. We have shown above that the emission externality induces a steepening of the NKPC. Holding monetary policy constant, the inflation response is still negative, but larger, which follows directly from Proposition 3. The natural output gap response under constant monetary is also still negative. Importantly, the NKPC does cross the origin, so it would be feasible to close the natural output gap and inflation simultaneously. However, the monetary policy rule (solid black line) indicates that this is no longer optimal. The central bank trades off some dis-inflation against maintaining a negative output gap, represented by the point Q. Since the monetary policy rule is also steeper than in the text book model, due to the larger weight on output stabilization, optimal natural output gap and inflation are even further away from zero.

Having derived optimal discretionary policy, we briefly revisit the implications of a "deflationary bias" in the presence of real externalities. It is a well-known result in monetary policy theory that there are no gains from commitment in the standard New Keynesian model with productivity shocks (Clarida et al., 1999). Since divine coincidence holds in this case, optimal monetary policy achieves the global minimum of its objective function even under discretion. By contrast, with emission externalities, divine coincidence is broken and the global minimum of the objective is not achieved by the optimal discretionary inflation and natural output gap responses, as prescribed in Proposition 5.

It comes natural to evaluate how large the gains of commitment are in the presence of real externalities. Therefore, we compute welfare (8) once under discretion by assuming that monetary policy implements the optimal interest rate derived under Eq. (24) and once under commitment, i.e. under Ramsey-optimal monetary policy. In both cases, we assume that the steady state emission tax is efficient. While the gains from commitment are monotonically increasing in γ, they are quantitatively small. Under the baseline value of $\gamma = 0.37$, they are merely 0.001% in consumption equivalents and another order of magnitude smaller when the externality parameter is reduced by 50% to $\gamma = 0.19$.

4.3. Demand and cost-push shocks

So far, we have provided an analytical characterization of optimal monetary policy in the case of productivity shocks. In this section, we demonstrate how the emission externality affects the conduct of monetary policy in the case of pure demand shocks and exogenous cost-push wedges, respectively. Conceptually, these results are derived in the same way as for productivity shocks. We begin with characterizing IS curve and NKPC and derive the loss function as second order approximation to household utility before solving the monetary policy problem with the LQ approach. As for Propositions 4 and 5, we restrict attention to the case of constant emission taxes, $\hat{\tau}_t = 0$.

As customary in the literature (Smets and Wouters, 2003), demand shocks are modeled as time-variation in intertemporal preferences obtained by modifying lifetime utility function in the following way:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \exp\{\rho_t\} \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right) \right],$$
$$\rho_t = \rho_B \rho_{t-1} + \sigma_B \epsilon_t^B, \quad \text{where } \epsilon_t^B \sim N(0,1).$$

Following Justiniano et al. (2013), we also add exogenous shocks to desired markups. This shock is typically used in the literature to break divine coincidence in the standard New Keynesian model, as that generates a wedge in the NKPC, making it impossible for the central bank to simultaneously stabilize inflation and output gap. We assume that the desired markup in (4) behaves as follows:

$$\mu_t = \left(\frac{\epsilon}{\epsilon-1}\right)^{1-\rho_M} \mu_{t-1}^{\rho_M} e^{\epsilon_t^M} , \quad \text{where } \epsilon_t^M \sim N(0,\sigma_M). \tag{28}$$

By following the steps of the proof for Proposition 2, it is straightforward to show how the time-variation in the discount factor and in desired markups both affect IS curve and NKPC in the log-linearized model:

$$x_t^n = \mathbb{E}_t[x_{t+1}^n] - \frac{r_t^s - \mathbb{E}_t[\pi_{t+1}]}{\sigma} + \underbrace{\frac{1}{\zeta}\mathbb{E}_t\Big[(1+\varphi)(a_{t+1}-a_t) - (\mu_{t+1}-\mu_t)\Big] + \frac{1}{\sigma}\mathbb{E}_t\left[\rho_t - \rho_{t+1}\right]}_{\equiv r_t^n/\sigma} , \tag{29}$$

$$\pi_t = \zeta\kappa x_t^n + \beta\mathbb{E}_t[\pi_{t+1}] + \mu_t . \tag{30}$$

Eqs. (29) and (30) generalize Eqs. (16) and (17). While demand shocks do not affect the NKPC at all, the shock to desired markups enters the NKPC directly as an exogenous cost-push wedge. The demand shock affects the IS curve and, hence, the natural rate of interest. However, the transmission of a demand shock $\mathbb{E}_t[\rho_t - \rho_{t+1}]$ is independent of the externality parameter γ, as the shock does not affect firms' marginal cost. The effect of markup shocks on the IS curve does depend on the externality via the auxiliary parameter ζ, similar to the TFP shock. Since the externality endogenously dampens the responses of actual and natural output levels, we observe a smaller response of x_t^n. As before, it turns out that the economically most interesting mechanisms are related to the central bank objective function, to which we turn next.

Proposition 6. *A second order approximation of the welfare function around the distorted steady state yields the following quadratic loss function:*

$$\mathcal{L} = -\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \frac{U_t - U}{U_c C}\right] \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\pi_t^2 + \omega_x\left(x_t^n - x_t^*\right)^2\right\}\right] + t.i.p. \tag{31}$$

where x_t^* is the target level of the natural output gap and ω_x is the weight on output stabilization:

$$\omega_x = \frac{\kappa}{\epsilon}(\zeta(1+\gamma)+\gamma) , \tag{32}$$

$$x_t^* = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma}a_t + \frac{1}{\zeta}\mu_t . \tag{33}$$

Proof. see Appendix B.1.

Proposition 6 yields two important insights. First, demand shocks do not affect the central bank objective function, irrespective of the emission externality. This is not surprising, as they did not enter the NKPC at all and also do not interact with the externality parameter in the IS curve. Consequently, they do not affect the efficient output gap beyond moving the natural rate of interest, which is given by Eq. (29) and imposing $x_t^n = \mathbb{E}_t[x_{t+1}^n] = 0$. Divine coincidence holds and the central bank is able to close the inflation and output gaps simultaneously by tracking the natural interest rate.

Second, cost-push shocks break divine coincidence already without the emission externality. This is reflected by the presence of a non-zero target level for the natural output gap x_t^* whenever $\mu_t \neq 0$. The reason is that markup shocks do not affect efficient output. Consider an expansionary markup shock that puts upward pressure on output downward pressure on inflation, since the optimal price is given by the markup times marginal cost. With sticky prices, non-adjusters have to satisfy this condition as well at the previously chosen higher price and can only do so by increasing their marginal cost, that is by demanding more labor. This pushes actual output above natural output and the natural gap x_t^n is positive, but smaller than the efficient output gap x_t^e.

How does the emission externality affect the over-reaction of the competitive equilibrium to expansionary shocks? In the baseline case with TFP shocks, the over-reaction of output is positively related to the emission externality and vanishes if $\gamma = 0$. With markup shocks, the discrepancy between natural and efficient output level emission externality *declines*, simply because the emission externality dampens the effect of the expansionary shock on y_t^n. The reason for this stark difference is that markup shocks do not affect firms' marginal cost, which are given by $mc_t = \frac{w_t}{A_t \Lambda_t}$, and also do not directly interact with the externality. Intuitively, emission damages do not affect firms ability to sell goods at a markup to households. By contrast, TFP shocks interact with the externality directly, since Λ_t affects productivity via the production function. Since firms do not take their effect on Λ_t on marginal costs into account when making their decisions, the discrepancy between competitive equilibrium and efficient allocation can be directly attributed to the externality. Such a "direct" discrepancy is not present for cost push shocks.

Lastly, it is worth noting that the weight on output stabilization ω_x is the same, irrespective of the shock, so it coincides with Eq. (22) in Proposition 4. Proposition 7 characterizes the optimal monetary policy response to each shock.

Proposition 7. *Under discretion, monetary policy responds optimally*

(i) to i.i.d. demand shocks by setting:

$$\pi_t^o = 0 , \tag{34}$$

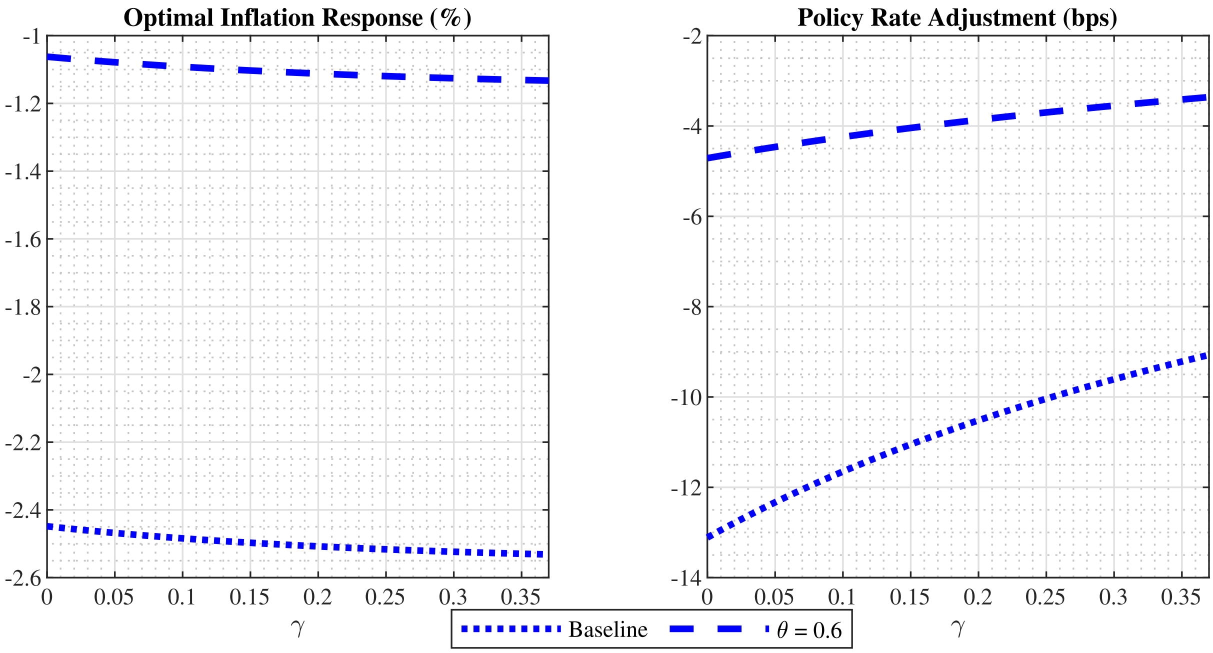

Fig. 7. Optimal Inflation and Interest Rate Adjustment, Markup Shock. This figure illustrates Proposition 5 by plotting the optimal response of inflation (left panel) and the interest rate adjustment term in Eq. (27) (right panel) to an expansionary markup shock ($\epsilon_t^M = -\sigma_M$) as functions of the externality parameter γ. The dotted line corresponds to the baseline Calvo parameter of $\theta = 0.8$, while the dashed line reflects the case of $\theta = 0.6$.

$$x_t^o = 0\,, \tag{35}$$

$$r_t^o = \frac{1}{\sigma}\rho_t = r_t^n. \tag{36}$$

(ii) to i.i.d. cost-push shocks by setting:

$$\pi_t^o = \omega_x \frac{1+\kappa}{\omega_x + \kappa^2\zeta^2}\mu_t\,, \tag{37}$$

$$x_t^o = \frac{\omega_x - \zeta^2\kappa}{\zeta(\omega_x + \zeta^2\kappa^2)}\mu_t\,, \tag{38}$$

$$r_t^o = r_t^n - \frac{\sigma}{\zeta}\frac{\omega_x - \zeta^2\kappa}{\omega_x + \zeta^2\kappa^2}\mu_t, \tag{39}$$

Proof. see Appendix B.2.

We have already seen from the loss function that demand shocks affect neither the welfare-relevant nor the natural output gap, such that it is optimal to track the natural interest rate. This perfectly stabilizes inflation and output gap. Fig. 7 illustrates optimal monetary policy in response to an expansionary cost push shock ($\epsilon_t^M = -\sigma_M$) for different externality and Calvo parameters. The left panel shows that the optimal inflation response is slightly more pronounced as γ increases. Again, this does not follow from the discrepancy between natural and efficient output gap like in the case of TFP shocks, but from the steepened NKPC. For a given change in the natural output gap, prices move more if γ is large and it is optimal to allow for slightly more disinflation, which goes hand in hand with a slightly more positive natural output gap. However, this effect is very small compared to the effects of changing the Calvo parameter. To implement these inflation responses, the adjustment parameter in the policy rate (39) is slightly smaller in absolute terms if γ is large.

5. Optimal monetary policy: Quantitative results

In this section, we show that our analytical results from the three-equation model carry over to the four-equation model where emission damages depend on the stock of emissions. We also demonstrate that the optimal monetary policy response differs from the standard model without an emission externality in a quantitatively relevant way. Furthermore, we provide several comparative statics exercises with respect to macroeconomic and climate parameters.

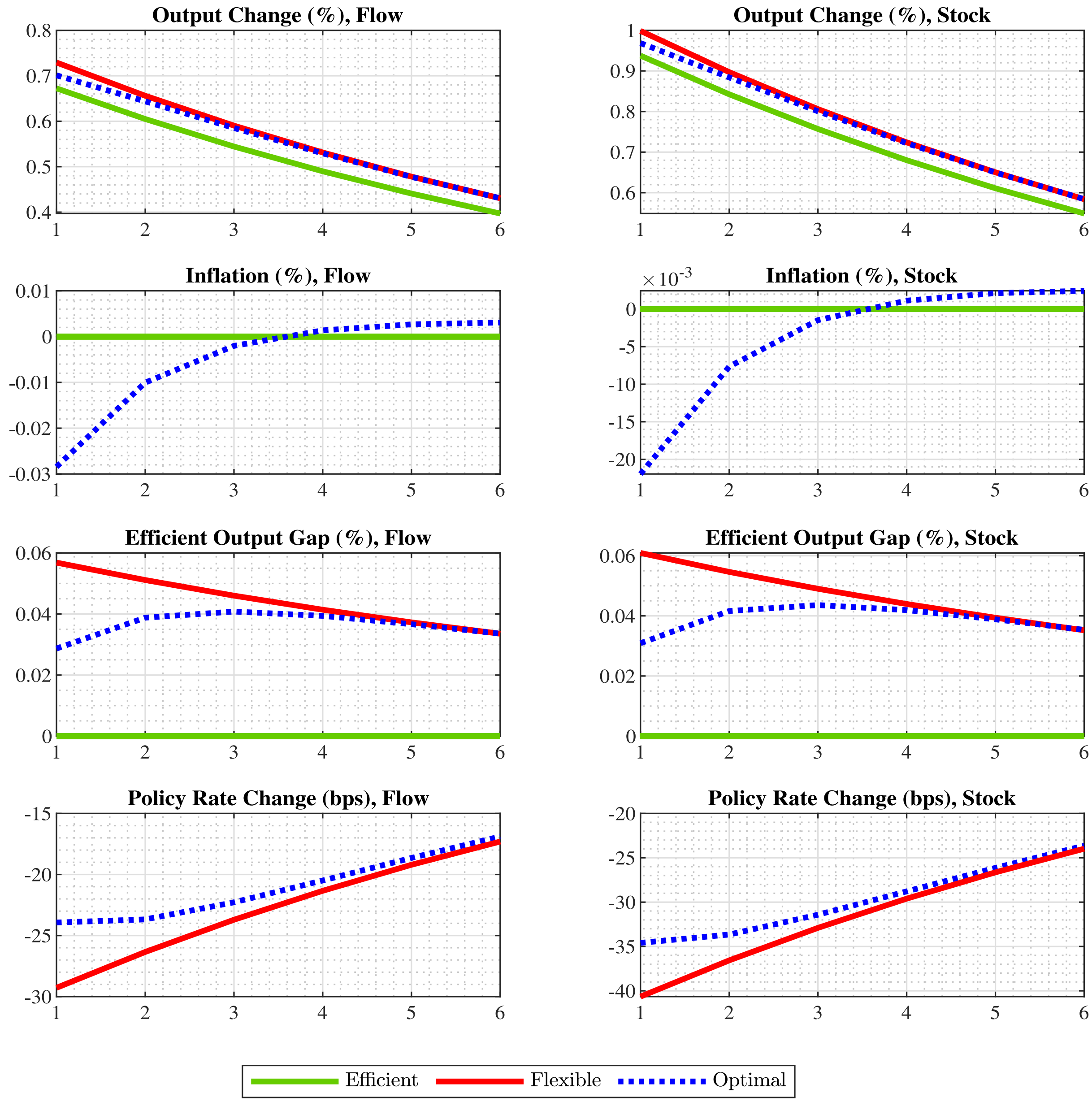

Fig. 8. Optimal Monetary Policy in Three- and Four-Equation Model. Impulse response to a positive one standard deviation shock to TFP over 6 quarters, using a second order approximation around the deterministic steady state. The left column refers to the three-equation model where damages depend on the flow of emissions. The right column refers to the four-equation model where damages depend on a persistent stock of emissions.

5.1. Impulse response functions

As a first step, we revisit the impulse response functions to a positive one standard deviation TFP shock in the model versions with a flow pollutant (three-equation, left column) and a stock pollutant (four-equation, right column). We have seen above in Fig. 2 that natural and efficient output gap are remarkably similar in both models. Fig. 8 shows that this similarity carries over to optimal monetary policy in both models. The red line refers to the natural output response and the natural rate of interest, respectively. The green line indicates the efficient allocation, while the dotted blue line indicates optimal monetary policy.

The first row shows that the economy with a flow pollutant exhibits a much smaller expansion due to the large effect of current output on contemporaneous emission damages. Output expands merely by around 0.75% in the three-equation model, compared to a 1% increase in the four-equation model. However, the *over-reaction* of output in the flexible price allocation is the key endogenous

Table 2

Optimal Monetary Policy: Macro and Welfare Effects: All moments are computed under optimal monetary policy in the presence of a constant carbon tax that renders the deterministic steady state efficient. Inflation volatility is annualized an expressed in percentage points, the policy rate is annualized and expressed in basis points. We express the welfare gain of optimal monetary policy in consumption equivalents
$gain^{CE,opt} \equiv \exp\{(1-\beta)(V^{opt} - V^{Taylor})\} - 1$
where V^{Taylor} refers to welfare (8) when central bank follows a Taylor rule (7) with inflation coefficient $\phi = 1.5$. The welfare gain of optimal monetary policy over the flexible price equilibrium, achieved by setting the coefficient ϕ in the Taylor rule (7) to a large number, is expressed relative to the welfare gain of the efficient allocation, achieved by Ramsey-optimal policy with time-varying interest rates *and* emission taxes, over the flexible price equilibrium:
relative gain $\equiv \frac{\exp\{(1-\beta)(V^{mp}-V^{flex})\}-1}{\exp\{(1-\beta)(V^{efficient}-V^{flex})\}-1}$.

	Main Results			Comparative Statics: Macro			Climate Block		
	$\gamma = 0$ (1)	Flow (2)	Stock (3)	$\theta = 0.6$ (4)	$\varphi = 0$ (5)	$\sigma = 2$ (6)	$\gamma = 0.17$ (7)	No Tax (8)	RoW (9)
Optimal impact response to σ_A shock									
Output change (%)	1.00	0.70	0.97	0.99	0.91	0.67	0.99	0.97	0.99
Inflation (%)	0	−0.03	−0.02	−0.03	−0.03	−0.02	−0.01	−0.02	−0.005
Eff. Output Gap x_t^e (%)	0	0.03	0.03	0.05	0.04	0.03	0.01	0.03	0.01
Policy rate (bps)	−41	−24	−34	−37	−27	−45	−38	−35	−39
Adjustment term (bps)	0	5	6	4	5	11	3	6	1
Volatility									
Output Dev. $\hat{y}_t$ (%)	2.29	1.65	2.25	2.26	2.20	1.57	2.27	2.25	2.28
Eff. Output Gap x_t^e (%)	0	0.11	0.12	0.13	0.23	0.11	0.06	0.12	0.02
Nat. Output Gap x_t^n (%)	0	0.03	0.03	0.01	0.10	0.03	0.02	0.03	0.01
Inflation (%)	0	0.03	0.02	0.04	0.04	0.03	0.01	0.024	0.005
Policy Rate (bps)	93	63	88	91	80	122	90	88	91
Welfare effect (CE, %)									
Gain over $\phi = 1.5$	0.07	0.03	0.05	0.01	0.04	0.13	0.06	0.05	0.06
Relative gain over flex price	0	0.14	0.11	0.10	0.20	0.10	0.10	–	0.10

variable for the conduct of monetary policy, which is reflected by the red line in the third row and is very similar in both economies. Consequently, the optimal monetary policy trade-off is solved in a similar fashion: Optimal monetary policy implements an output expansion between natural and efficient level, such that the efficient output gap is always positive, see the third row of Fig. 8.

Notably, this holds for any degree of the price rigidity, not only for values near our baseline θ. Why does the central bank never expands output by less than the efficient allocation? Consider a choice between increasing output by 0.01 percentage points more or 0.01 percentage points less than the efficient allocation, i.e. the green line in the top panel of Fig. 8. Loosely speaking, both choices imply the same deviation from the first best output expansion. However, increasing output by less than the efficient allocation entails a worse outcome on price stabilization, as it implies a larger deviation from the flexible price equilibrium.

As the optimal output response falls short of the natural output response, inflation has to fall in response to an expansionary TFP shock, which follows from the NKPC and is shown in the second row of Fig. 8. To implement this, the optimal interest rate cut is smaller (in absolute terms) than what would be necessary to implement the flexible price equilibrium. Comparing the baseline four-equation to the simplified three-equation model, it turns out that both economies differ in their natural rate, but the discrepancy between the natural rate and the optimal monetary policy rate (the adjustment term in (27)) is quite similar at around 6 basis points. For large shock realizations, the discrepancy between natural and optimal rate is quantitatively relevant, given that we are considering a comparatively small shock size and that the typical monetary policy rate notch is 25 basis points.

5.2. Macro and welfare effects

As a second step, we explore the quantitative implications of pro-cyclical emissions for optimal monetary policy, macroeconomic aggregates and welfare. In the first panel, we display the impulse response of key macro variables to a one standard deviation TFP shock implied by optimal monetary policy. The second panel considers the time series volatility of macroeconomic aggregates under optimal monetary policy, while the last panel displays the welfare implications.

The standard New Keynesian model without externalities is shown as a reference point in column (1) of Table 2. Here, monetary policy can implement the efficient allocation by tracking the natural rate of interest. The optimal interest rate cut after an expansionary productivity shock is 41 basis points, resulting in a natural interest rate volatility of 93 basis points. Inflation, efficient and natural output gap are zero in all states and the volatility of output coincides with a real business cycle model. The welfare gain relative to the case where monetary policy follows a simple Taylor rule (7) with a coefficient $\phi = 1.5$ amounts to 0.07% in consumption equivalents, which is a typical magnitude in a representative agent model of the business cycle.

Column (2) considers the three-equation model – with a flow pollutant – that we discussed analytically above. Here, output volatility is considerably smaller, which we have shown formally in Proposition 3: the standard deviation of output declines from 2.29% to 1.65%, as emission damages immediately dampen output responses. Monetary policy does not implement the efficient allocation: Inflation falls to −0.03% in response to the shock, while output expands 0.03 percentage points above the efficient allocation. The central bank achieves this by reacting less aggressively to the shock. The interest rate cut of 24bps is 5bps smaller than the natural rate decline, which corresponds to the adjustment term in Proposition 5.

Inflation and both output gaps exhibit a positive volatility, as the middle panel shows, while the interest rate volatility declines to 63bps, which represents both the smaller volatility of the natural interest rate and the adjustment term. The welfare gain of optimal monetary policy over the Taylor rule with $\phi = 1.5$ is smaller than in the standard model, as optimal monetary policy achieves neither price stability nor does it close the efficient output gap. The last row computes the welfare gain of optimal monetary policy over a policy that ignores pro-cyclical emissions by simply tracking the natural interest rate. As the lack of pro-cyclical emission taxes is the ultimate reason why monetary policy addresses two inefficiencies at once, we compute the welfare gain relative to the difference between efficient allocation and flexible price allocation. It turns out that monetary policy achieves 14% of the maximally possible welfare gain, which suggests that the tension between pro-cyclical emissions and nominal rigidities is substantial.

The four-equation model with the stock pollutant is shown in column (3). The impact responses of output and natural rate as well as their time series volatilities are much closer to the standard model. However, the responses and volatilities of inflation and efficient output gap are very similar to the three-equation model. This is also reflected in the adjustment term of 6bps, which is almost identical to the three-equation model.

Comparative statics: Macro parameters We then conduct several comparative statics exercises in the four-equation model with respect to key macro parameters. First, we reduce the Calvo parameter to $\theta = 0.6$, which is at the lower bound of the parameter space, as it implies an expected price duration of slightly more than half a year. With smaller price rigidities it becomes more costly to deviate from the flexible price allocation — the interest rate adjustment term of 4bps is slightly smaller than in the baseline. Consequently, the efficient output gap response to a shock is larger (0.05%) and the relative gain of optimal monetary policy over the flexible price allocation is slightly diminished.

In column (4) and (5), we consider different parameters in the utility function. When we consider linear labor disutility ($\phi = 0$), the gains of output gap stabilization decline. Consequently, the efficient output gap is much more volatile (0.23%). The opposite picture emerges when we increase households risk aversion to $\sigma = 2$. If the gains of macroeconomic stabilization are higher, the weight ω_x on the natural output gap increases while its target level declines in absolute terms x_t^*. Consequently, optimal monetary policy deviates more aggressively from the natural rate. The interest rate adjustment term now amounts to 11bps, which is almost twice the size of the adjustment term in the baseline calibration. Notably, the volatility of inflation and efficient output gap hardly change compared to the baseline, so that the relative gain over replicating the flexible price allocation is also similar at 10%.

Comparative statics: Climate policy Next, we modify parts of the climate block. The welfare losses of pro-cyclical emissions are positively related to the severity of the emission externality. Since the social cost of carbon are subject to considerable uncertainty, column (7) displays the results of setting $\gamma = 0.17$, which reduces emission damages by 50%. Here, optimal monetary policy brings the economy is much closer to the flexible price allocation, which is reflected in the smaller adjustment term of 3bps. Time series volatilities and the welfare gain of optimal monetary policy almost coincides with the standard model (0.06%). Column (8) demonstrates that our results do not depend on the steady state emission tax and we focus on the extreme case of zero taxes. We have seen analytically in Proposition 4 that the central bank loss function completely separates business cycle stabilization objectives from long run inefficiencies.

Throughout the quantitative application to carbon dioxide emissions and climate change, we have implicitly interpreted the model as representing the world economy, similar to the quantitative analysis by Sahuc et al. (2024). To the extent that cyclical variations in output and emissions abroad are beyond the control of any domestic central bank, it is reasonable to assume that rest-of-the-world emissions are not considered in the over-reaction relative to the efficient allocation.[15] We use a similar damage function $\Lambda_t = \exp\{-\gamma \frac{y_t}{E} - \gamma^{ROW}\}$ and re-parameterize it to $\gamma = 0.07$ and $\gamma = 0.3$. These values are reasonable for large economies like China, the US, and the euro area. Note that damages associated with rest-of-the-world emissions still enter the IS-equation and the NKPC, but they do not affect the efficient output *gap*. The results of this extension, shown in column (9) are very similar to the case of reducing γ, but even more pronounced: Optimal monetary policy behaves almost as in the standard model.

6. Conclusion

In this paper, we have explored the interactions between pro-cyclical emissions, real externalities and nominal rigidities in an environmental New Keynesian model and study its implications for optimal monetary policy. Our analysis is motivated by the strong pro-cyclicality of emissions in the data: Firms do not take into account that expanding production during a boom increases socially harmful emissions, which inflicts macroeconomically relevant damages on the wider economy. The optimal emission tax is pro-cyclical and, importantly, a constant tax that is efficient in steady state does not implement first best. Under a constant tax, output over-reacts to productivity shocks in the flexible price equilibrium, which counteracts the typical under-reaction of output in the sticky price equilibrium of the standard New Keynesian model.

We analytically solve for optimal monetary policy as a second-best solution to a welfare maximization problem, maintaining the assumption of a constant emission tax. From a methodological perspective, we demonstrate that an additional condition is necessary to ensure that the central bank loss function appropriately internalizes the over-reaction of output to productivity shocks. This condition links the output level to emission damages. To eliminate the level of emissions and economic activity in the loss

[15] This is not obvious for large central banks, whose policies also transmit to the rest-of-the-world.

function, we exploit and additional second order approximation around of the relationship between output and emission damages. This facilitates a clean separation between business cycle stabilization objectives and steady state distortions and extends the analysis of Benigno and Woodford (2005) to competitive equilibria that are inefficient due to real externalities.

Building on this methodological contribution, we uncover two main analytical results. First, closing the natural output gap is not optimal from a utilitarian welfare perspective, even though this would be feasible: divine coincidence is broken even for TFP shocks. Second, to tackle this dynamic inefficiency, the central bank optimally targets a non-zero natural output gap. These results imply that the optimal inflation volatility is unambiguously larger and the welfare gain of optimal monetary policy is smaller than in the absence of emission externalities. Quantitatively, the optimal monetary policy response to a one standard deviation productivity differs by around 6 basis points from the optimal response in a counterfactual economy without pro-cyclical emissions. These results are robust to reasonable variations in macro parameters, but critically depend on the severity of the emission externality and the elasticity of emissions to output.

Our analysis abstracts from technological choice, for example by allowing for costly emission abatement, technological change or shifts in the sectoral composition, as conventional monetary policy is traditionally concerned with aggregate variables such as inflation and the output gap, rather than the sectoral composition of output. Recent research on monetary policy in the presence of sectoral shocks has contested this view (Guerrieri et al., 2021) and the analysis of efficient and natural output gaps with a disaggregated sectoral structure or endogenous technological choices appears to be a promising avenue for future research. Furthermore, there is evidence also that socially harmful emissions also have a direct effect on macroeconomic volatility and inflation through a disaster risk channel (Cantelmo et al., 2024), from which we abstract in our analysis. Lastly, carbon taxation can also induce inflation by increasing energy prices, which has been subject to recent discussions about targeting core and headline inflation (Olovsson and Vestin, 2023). Exploring the interactions between these additional channels, nominal rigidities, and its implications for monetary policy is left for future research.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Francesco Giovanardi reports financial support was provided by European Commission, Next Generation EU. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proofs for productivity shocks

This section contains all proofs for the case of productivity shocks omitted in the main text.

A.1. Proof of Proposition 1

The aggregate production function can be written as $y_t = A_t \Lambda_t n_t$, while the goods market clearing condition is given by $y_t = c_t$.

Efficient allocation The planner problem is

$$\max_{c_t,n_t,y_t,\Lambda_t,E_t} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right] \quad \text{s.t.}$$

$$c_t = y_t \,, \qquad (\lambda_t)$$

$$y_t = A_t \Lambda_t n_t \,, \qquad (\mu_t)$$

$$\Lambda_t = \exp\left\{ -\gamma \frac{E_t}{E} \right\} \,, \qquad (\nu_t)$$

$$E_t = y_t + (1-\delta) E_{t-1} \,. \qquad (\psi_t)$$

Setting up the Lagrangian

$$\max_{c_t,n_t,y_t,\Lambda_t,E_t} \sum_{t=0}^{\infty} \beta^t \Bigg[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} + \lambda_t \Big(y_t - c_t \Big) + \mu_t \Big(A_t \Lambda_t n_t - y_t \Big) + \nu_t \Big(\exp\left\{ -\gamma \frac{E_t}{E} \right\} - \Lambda_t \Big) + \psi_t \Big(E_t - y_t - (1-\delta) E_{t-1} \Big) \Bigg]$$

and taking FOCs yields

$$\lambda_t = c_t^{-\sigma} \tag{A.40}$$

$$\mu_t A_t \Lambda_t = n_t^{\varphi} \tag{A.41}$$

$$\lambda_t - \mu_t - \psi_t = 0 \tag{A.42}$$

$$\mu_t A_t n_t = \nu_t \tag{A.43}$$

$$-\nu_t \frac{\gamma}{E}\Lambda_t + \psi_t - \beta(1-\delta)\psi_{t+1} = 0 \tag{A.44}$$

Combining (A.43) and (A.44):

$$\psi_t = \beta(1-\delta)\psi_{t+1} + \mu_t \frac{\gamma}{E} y_t$$

With (A.42):

$$\lambda_t = \beta(1-\delta)\big(\lambda_{t+1} - \mu_{t+1}\big) + \mu_t \frac{\gamma}{E} y_t + \mu_t$$

With $\delta = 1$, we have $E = y$ and this expression collapses to the expression for MPN^e that we have derived in the main text. With $\delta < 1$, we can re-arrange to

$$\lambda_t - \mu_t\big(1 + \frac{\gamma}{E} y_t\big) = \beta(1-\delta)\big(\lambda_{t+1} - \mu_{t+1}\big)$$

Using (A.40) and (A.41):

$$c_t^{-\sigma} - \frac{n_t^{\varphi}}{A_t \Lambda_t}\big(1 + \frac{\gamma}{E} y_t\big) = \beta(1-\delta)\big(c_{t+1}^{-\sigma} - \frac{n_{t+1}^{\varphi}}{A_{t+1}\Lambda_{t+1}}\big) \tag{A.45}$$

Exploiting $c_t = y_t = A_t \Lambda_t n_t$, the efficient output is given by the solution y_t^e to:

$$y_t^{-\sigma} - \frac{y_t^{\varphi}}{(A_t \Lambda_t)^{1+\varphi}}\big(1 + \frac{\gamma}{E} y_t\big) = \beta(1-\delta)\big(y_{t+1}^{-\sigma} - \frac{y_{t+1}^{\varphi}}{(A_{t+1}\Lambda_{t+1})^{1+\varphi}}\big). \tag{A.46}$$

Competitive equilibrium Next, we derive the natural level of output consistent with flexible prices. The relevant equilibrium conditions are:

- Euler equation and labor supply condition,

$$c_t^{-\sigma} = \beta r_t^s \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{\pi_{t+1}}\right], \tag{A.47}$$

$$n_t^{\varphi} = w_t c_t^{-\sigma}. \tag{A.48}$$

- emission damage function:

$$\Lambda_t = \exp\left(-\gamma \frac{E_t}{E}\right), \tag{A.49}$$

- aggregate production function:

$$\Delta_t y_t = A_t \Lambda_t n_t, \tag{A.50}$$

  where Δ_t is the price dispersion.
- labor demand:

$$(1-\tau^n) w_t = mc_t A_t \Lambda_t, \tag{A.51}$$

- good market clearing:

$$y_t = c_t. \tag{A.52}$$

- Optimal pricing

$$p_t^* = \frac{\mu}{1-\tau_t^c}\frac{\xi_{1,t}}{\xi_{2,t}}, \tag{A.53}$$

  where $\mu \equiv \frac{\epsilon}{\epsilon-1}$ and

$$\xi_{1,t} = mc_t y_t + \beta\theta \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon} \xi_{1,t+1}, \tag{A.54}$$

$$\xi_{2,t} = y_t + \beta\theta \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon-1} \xi_{2,t+1}. \tag{A.55}$$

- Inflation:

$$1 = (1-\theta)(p_t^*)^{1-\epsilon} + \theta \pi_t^{\epsilon-1}. \tag{A.56}$$

- Price dispersion:

$$\Delta_t = (1-\theta)(p_t^*)^{-\epsilon} + \theta \pi_t^{\epsilon} \Delta_{t-1}. \tag{A.57}$$

Eqs. (A.47) to (A.57) characterize the competitive equilibrium for the eleven endogenous variables $\{\Delta_t, y_t, \Lambda_t, n_t, w_t, mc_t, c_t, p_t^*, \xi_{1,t}, \xi_{2,t}, \pi_t\}$.

If prices are flexible, we normalize the price level such that $\Delta_t = \pi_t = p_t^* = 1$ and $\frac{\xi_{1,t}}{\xi_{2,t}} = mc_t$. In this case, (A.53) simplifies to $p_t^* = \frac{\mu}{(1-\tau_t^c)} mc_t$. Marginal costs are simply given by $mc_t = \frac{(1-\tau^n)w_t}{A_t\Lambda_t}$. If the labor subsidy $\tau^n = \frac{1}{\epsilon}$ corrects for the steady state monopolistic distortion, using Eqs. (A.48) and (A.51) gives:

$$1 = \frac{\mu}{1-\tau_t^c} mc_t = \frac{1}{1-\tau_t^c} \underbrace{\mu(1-\tau^n)}_{=1} \frac{w_t}{A_t\Lambda_t} = \frac{n_t^{\varphi} c_t^{\sigma}}{(1-\tau_t^c)A_t\Lambda_t} . \tag{A.58}$$

Using goods market clearing (A.52) and the production function (A.50) to eliminate c_t and n_t, we obtain

$$1 = \frac{y_t^{\sigma+\varphi}}{(1-\tau_t^c)(A_t\Lambda_t)^{1+\varphi}}.$$

Solving for y_t yields the natural output level (9). □

A.2. Proof of Lemma 1

Lemma 1 is a straightforward simplification of Proposition 1. Log-linearizing the natural output level (9) around the deterministic steady state gives

$$(\sigma+\varphi)\hat{y}_t^n = (1+\varphi)a_t - (1+\varphi)\gamma\hat{y}_t^n - \frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c ,$$

where hats indicate log-deviations from steady state. Re-arranging for $\hat{y}_t^n$ yields Eq. (13). Log-linearizing the efficient output level (11) and noticing that $\hat{\lambda}_t = -\gamma\hat{y}_t$ yields

$$\begin{aligned} &(\sigma+\varphi)\hat{y}_t^e = (1+\varphi)a_t - (1+\varphi)\gamma\frac{y^e}{y}\hat{y}_t^e - \frac{\gamma}{1+\gamma}\hat{y}_t^e \\ \Leftrightarrow\ &\left[\sigma+\varphi+(1+\varphi)\gamma\frac{y^e}{y} + \frac{\gamma}{1+\gamma}\right]\hat{y}_t^e = (1+\varphi)a_t. \end{aligned} \tag{A.59}$$

Re-arranging for $\hat{y}_t^e$, we arrive at Eq. (14). □

A.3. Proof of Proposition 2

The log-linearized equilibrium conditions in the simplified model ($\delta = 1$) are given by

- Euler Eq. (A.47):

$$\sigma\hat{c}_t = \sigma\hat{c}_{t+1} - (\hat{r}_t^s - \hat{\pi}_{t+1}). \tag{A.60}$$

- Optimal labor supply (A.48):

$$\hat{w}_t = \varphi\hat{n}_t + \sigma\hat{c}_t. \tag{A.61}$$

- Emission damages (A.49):

$$\hat{\Lambda}_t = -\gamma\hat{y}_t. \tag{A.62}$$

- Production function (A.50):

$$\hat{\Delta}_t + \hat{y}_t = a_t - \gamma\hat{y}_t + \hat{n}_t. \tag{A.63}$$

- Labor demand (A.51):

$$\hat{w}_t = \widehat{mc}_t + a_t - \gamma\hat{y}_t. \tag{A.64}$$

- Optimal pricing (A.53), (A.54), and (A.55):

$$\hat{p}_t^* = \frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c + \hat{\xi}_{1t} - \hat{\xi}_{2t} \tag{A.65}$$

$$\widehat{\xi_{1,t}} = (1-\theta\beta)\widehat{mc}_t + (1-\theta\beta)\hat{y}_t - \theta\beta\sigma\hat{c}_{t+1} + \theta\beta\sigma\hat{c}_t + \epsilon\theta\beta\hat{\pi}_{t+1} + \theta\beta\hat{\xi}_{1,t+1} \tag{A.66}$$

$$\widehat{\xi_{2,t}} = (1-\theta\beta)\hat{y}_t - \theta\beta\sigma\hat{c}_{t+1} + \theta\beta\sigma\hat{c}_t + (\epsilon-1)\theta\beta\hat{\pi}_{t+1} + \theta\beta\hat{\xi}_{2,t+1} , \tag{A.67}$$

  where we assumed no steady state inflation, $\bar{\pi} = 1$.
- Inflation (A.56):

$$0 = (1-\epsilon)(1-\theta)\hat{p}_t^* + \theta(\epsilon-1)\hat{\pi}_t \ \Leftrightarrow\ \hat{p}_t^* = \frac{\theta}{1-\theta}\hat{\pi}_t . \tag{A.68}$$

- Price dispersion (A.57):

$$\hat{\Delta}_t = -\epsilon(1-\theta)\hat{p}_t^* + \theta\epsilon\hat{\pi}_t + \theta\hat{\Delta}_{t-1} \;\Leftrightarrow\; \hat{\Delta}_t = \theta\hat{\Delta}_{t-1} \Leftrightarrow \hat{\Delta}_t = 0\,.$$

- Market clearing (A.52):

$$\hat{c}_t = \hat{y}_t\,. \tag{A.69}$$

As before, we define the natural output gap as

$$x_t^n = \hat{y}_t - \hat{y}_t^n = \hat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c\right],$$

and efficient output gap:

$$x_t^e = \hat{y}_t - \hat{y}_t^e = \hat{y}_t - \frac{1}{\widetilde{\zeta}+\widetilde{\gamma}}\left[(1+\varphi)a_t\right].$$

We first derive the log-linearized NKPC. Subtracting the auxiliary terms in the optimal pricing condition, (A.66) and (A.67), from each other, we have

$$\hat{\xi}_{1t} - \hat{\xi}_{2t} = (1-\theta\beta)\widehat{mc}_t + \theta\beta\pi_{t+1} + \theta\beta(\hat{\xi}_{1,t+1} - \hat{\xi}_{2,t+1}).$$

Plugging this condition and the definition of inflation (A.68) into the expression for the optimal price (A.65), we get an expression for marginal costs:

$$\begin{aligned}
\frac{\theta}{1-\theta}\pi_t &= \frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c + (1-\theta\beta)\widehat{mc}_t + \theta\beta\left(\pi_{t+1} + \frac{\theta}{1-\theta}\pi_{t+1} - \frac{\tau^c}{1-\tau^c}\hat{\tau}_{t+1}^c\right)\\
\Leftrightarrow \pi_t &= \underbrace{\frac{(1-\theta\beta)(1-\theta)}{\theta}}_{=\kappa}\widehat{mc}_t + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\left(\hat{\tau}_t - \theta\beta\hat{\tau}_{t+1}^c\right).
\end{aligned} \tag{A.70}$$

Combining labor supply (A.61), production function (A.63), and labor demand (A.64), we get a second condition linking output to marginal costs:

$$\begin{aligned}
\widehat{mc}_t &= \hat{w}_t - a_t + \gamma\hat{y}_t = \varphi\hat{n}_t + \sigma\hat{c}_t - a_t + \gamma\hat{y}_t = \varphi(\hat{y}_t - a_t + \gamma\hat{y}_t) + \sigma\hat{y}_t - a_t - \gamma\hat{y}_t\\
&= \underbrace{[\sigma + \varphi + (1+\varphi)\gamma]}_{=\zeta}\hat{y}_t - (1+\varphi)a_t.
\end{aligned}$$

Plugging this condition into Eq. (A.70):

$$\begin{aligned}
\pi_t &= \kappa\zeta\left[\underbrace{y_t - \frac{1+\varphi}{\zeta}a_t + \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c}_{=x_t^n} - \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c\right] + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\left(\hat{\tau}_t^c - \theta\beta\hat{\tau}_{t+1}^c\right)\\
&= \kappa\zeta x_t^n + \beta\pi_{t+1} - \kappa\frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c + \frac{\kappa}{1-\theta\beta}\frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c - (1-\theta)\beta\frac{\tau^c}{1-\tau^c}\hat{\tau}_{t+1}^c\\
&= \kappa\zeta x_t^n + \beta\pi_{t+1} + (1-\theta)\beta\frac{\tau^c}{1-\tau_c}(\hat{\tau}_t^c - \hat{\tau}_{t+1}^c)\,,
\end{aligned}$$

which is the NKPC (17) in Proposition 2.

To get the dynamic IS-equation (16), start from the linearized Euler equation Eq. (A.60) and impose market clearing (A.69) to get:

$$\begin{aligned}
&\hat{y}_t = \hat{y}_{t+1} - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \;\Leftrightarrow\\
&\hat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c\right] + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c\right] =\\
&\hat{y}_{t+1} - \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\hat{\tau}_{t+1}^c\right] + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\hat{\tau}_{t+1}^c\right] - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \;\Leftrightarrow\\
&x_t^n + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\hat{\tau}_t^c\right] = x_{t+1}^n + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\hat{\tau}_{t+1}^c\right] - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) \;\Leftrightarrow\\
&x_t^n = x_{t+1}^n - \frac{1}{\sigma}(r_t^s - \pi_{t+1}) + \frac{1}{\zeta}\left[(1+\varphi)(a_{t+1} - a_t) - \frac{\tau^c}{1-\tau^c}(\hat{\tau}_{t+1}^c - \hat{\tau}_t^c)\right] \quad \square
\end{aligned}$$

A.4. Proof of Proposition 3

The proof uses the method of undetermined coefficients. Guess a linear policy function for $x_t^n = \Theta_{xa} a_t$ and $\pi_t = \Theta_{\pi a} a_t$, and impose equilibrium consistency in Eq. (16), Eq. (17), and Eq. (7), together with $\mathbb{E}_t[a_{t+1}] = \rho_a a_t$ and $\tau_t^c = 0$ to get:

$$\Theta_{xa} a_t = \Theta_{xa} \rho_a a_t - \frac{\phi \Theta_{\pi a} a_t - \Theta_{\pi a} \rho_a a_t}{\sigma} + \frac{1}{\zeta}\left[(1+\varphi)(\rho_a a_t - a_t)\right],$$

$$\Theta_{\pi a} a_t = \zeta \kappa \Theta_{xa} a_t + \beta \Theta_{\pi a} \rho_a a_t .$$

For the guess to be correct, the last two equations have to hold for each $a_t \in \mathcal{R}$. Hence, imposing $a_t = 1$ and solving the system of the two equations into the two unknowns, $\Theta_{\pi a}$ and Θ_{xa} yields the coefficients of the policy functions:

$$\Theta_{xa} = \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a) + \zeta\kappa(\phi - \rho_a)} \cdot (\rho_a - 1)\,, \tag{A.71}$$

$$\Theta_{\pi a} = \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a) + \zeta\kappa(\phi - \rho_a)} \cdot (\rho_a - 1)\,. \quad \square \tag{A.72}$$

A.5. Proof of Proposition 4

As a first step, we take a **second order approximation** of the welfare objective, i.e. households period utility function U_t:

$$U_t - U \approx c^{1-\sigma}\left\{\frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right]\right\}.$$

Re-arranging yields

$$\frac{U_t - U}{U_c c} = \frac{U_t - U}{c^{1-\sigma}} \approx \frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right].$$

For a generic variable x, up to second order, it holds that $\frac{x_t - x}{x} = \hat{x}_t + \frac{\hat{x}_t^2}{2}$ with $\hat{x}_t \equiv \log x_t - \log x$. Also, from the steady state condition for natural output Eq. (11) and market clearing, the following condition holds:

$$\frac{n^{1+\varphi}}{c^{1-\sigma}} = y^{1+\varphi}\frac{y^{\sigma-1}}{(A\Lambda)^{1+\varphi}} = \frac{y^{\sigma+\phi}}{(A\Lambda)^{1+\varphi}} = \frac{(1-\tau^c)(A\Lambda)^{1+\varphi}}{(A\Lambda)^{1+\varphi}} = 1 - \tau^c$$

Hence, we can re-write the second order approximation of the welfare function as:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \hat{c}_t + \frac{\hat{c}_t^2}{2} - \frac{\sigma}{2}\hat{c}_t^2 - (1-\tau^c)\left[\hat{n}_t + \frac{\hat{n}_t^2}{2} + \frac{\varphi}{2}\hat{n}_t^2\right].$$

In order to **express the loss function in terms of the output gap** x_t^n and inflation π_t, we first make use of the market clearing condition $\hat{c}_t = \hat{y}_t$ and the production function $\hat{n}_t = \hat{y}_t + \hat{\Delta}_t - a_t - \hat{\Lambda}_t$:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \hat{y}_t + \frac{1-\sigma}{2}\hat{y}_t^2 - (1-\tau)\left[\hat{y}_t + \hat{\Delta}_t - a_t - \hat{\Lambda}_t + \frac{1+\varphi}{2}\left(\hat{y}_t + \hat{\Delta}_t - a_t - \hat{\Lambda}_t\right)^2\right]$$

Define $\Phi \equiv (1-\tau)(1+\gamma) - 1$, so that:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \hat{y}_t + \frac{1-\sigma}{2}\hat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\left[\hat{y}_t + \hat{\Delta}_t - a_t - \hat{\Lambda}_t + \frac{1+\varphi}{2}\left(\hat{y}_t + \hat{\Delta}_t - a_t - \hat{\Lambda}_t\right)^2\right]$$

The newly defined parameter Φ is a measure of the steady state inefficiency: it takes the maximum value γ if $\tau = 0$ and it decreases to 0 as τ approaches the efficient level $\frac{\gamma}{1+\gamma}$. Throughout the proof, we consider the general case of a sub-optimal steady state carbon tax, $\tau^c \leq \tilde{\gamma}$, i.e. $\Phi \geq 0$. Eliminating all terms independent of policy and of order higher than two, we then obtain:

$$\begin{aligned}
\frac{U_t - U}{c^{1-\sigma}} \approx & \hat{y}_t + \frac{1-\sigma}{2}\hat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\left[\hat{y}_t + \hat{\Delta}_t - \hat{\Lambda}_t + \frac{1+\varphi}{2}\hat{y}_t^2 + \frac{1+\varphi}{2}\hat{\Lambda}_t^2\right.\\
& \left. - (1+\varphi)\hat{y}_t a_t - (1+\varphi)\hat{y}_t\hat{\Lambda}_t + (1+\varphi)\hat{\Lambda}_t a_t\right] + t.i.p.\\
\approx & \frac{\gamma - \Phi}{1+\gamma}\hat{y}_t + \frac{1+\Phi}{1+\gamma}\hat{\Lambda}_t - \frac{\hat{y}_t^2}{2}\left[-1 + \sigma + \frac{1+\Phi}{1+\gamma}(1+\varphi)\right] - \frac{1+\Phi}{1+\gamma}\hat{\Delta}_t\\
& - \frac{(1+\Phi)(1+\varphi)}{2(1+\gamma)}\hat{\Lambda}_t^2 + \frac{1+\Phi}{1+\gamma}(1+\varphi)(\hat{y}_t a_t + \hat{y}_t\hat{\Lambda}_t - \hat{\Lambda}_t a_t) + t.i.p.
\end{aligned}$$

We are then ready to **evaluate the loss function**:

$$\begin{aligned}
\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{ & \frac{1}{2}\left\{-1 + \sigma + \frac{1+\Phi}{1+\gamma}(1+\varphi)\right\}y_t^2 + \frac{1+\Phi}{1+\gamma}\hat{\Delta}_t + \frac{(1+\Phi)(1+\varphi)}{2(1+\gamma)}\Lambda_t^2 -\\
& - \frac{1+\Phi}{1+\gamma}(1+\varphi)(\hat{y}_t a_t + \hat{y}_t\hat{\Lambda}_t - \hat{\Lambda}_t a_t) + \frac{\Phi - \gamma}{1+\gamma}\hat{y}_t - \frac{1+\Phi}{1+\gamma}\hat{\Lambda}_t\Bigg\}\Bigg].
\end{aligned}$$

We first substitute the price dispersion term by an expression related to the NKPC via inflation. The discounted sum of log price dispersions is given by $\sum_{t=0}^{\infty}\beta^t\widehat{\Delta}_t \approx \frac{\epsilon}{2\kappa}\sum_{t=0}^{\infty}\beta^t\pi_t^2$, with the auxiliary parameter $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta}$ governing the slope of the NKPC. Therefore, the loss function is given by

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(-1+\sigma+\frac{1+\Phi}{1+\gamma}(1+\varphi)\Big)\widehat{y}_t^2+\frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\pi_t^2+\frac{(1+\Phi)(1+\varphi)}{2(1+\gamma)}\widehat{\Lambda}_t^2\right.\right.$$
$$\left.\left.-\frac{1+\Phi}{1+\gamma}(1+\varphi)(\widehat{y}_t a_t+\widehat{y}_t\widehat{\Lambda}_t-\widehat{\Lambda}_t a_t)+\frac{\Phi-\gamma}{1+\gamma}\widehat{y}_t-\frac{1+\Phi}{1+\gamma}\widehat{\Lambda}_t\right\}\right]. \tag{A.73}$$

As next step, we **eliminate the linear terms** $\frac{\Phi-\gamma}{1+\gamma}\widehat{y}_t$ and $\frac{1+\Phi}{1+\gamma}\widehat{\Lambda}_t$ in (A.73). To do so, we take a second order approximation of both the optimal pricing condition (A.53) and of the relationship between pollution and economic activity (A.49).

Starting from the latter, the second order approximation of Eq. (A.49) reads:

$$\widehat{\Lambda}_t+\frac{1}{2}\widehat{\Lambda}_t^2=-\gamma\widehat{y}_t-\frac{1}{2}\gamma(1-\gamma)\widehat{y}_t^2 \Leftrightarrow \widehat{\Lambda}_t=-\gamma\widehat{y}_t-\frac{1}{2}\gamma(1-\gamma)\widehat{y}_t^2-\frac{1}{2}\widehat{\Lambda}_t^2\,. \tag{A.74}$$

Plugging this into (A.73), we get:

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(-1+\sigma+\frac{1+\Phi}{1+\gamma}(1+\varphi+\gamma(1-\gamma))\Big)\widehat{y}_t^2+\frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\pi_t^2\right.\right.$$
$$\left.\left.+\frac{(1+\Phi)(2+\varphi)}{2(1+\gamma)}\widehat{\Lambda}_t^2-\frac{1+\Phi}{1+\gamma}(1+\varphi)(\widehat{y}_t a_t+\widehat{y}_t\widehat{\Lambda}_t-\widehat{\Lambda}_t a_t)+\Phi\widehat{y}_t\right\}\right]\,. \tag{A.75}$$

For what concerns the pricing condition, to eliminate the linear term Φy_t, we exploit that a second order approximation of Eq. (A.53) leads to the following (extended) NKPC:

$$\pi_t+\frac{\epsilon-1}{2(1-\theta)}\pi_t^2+\frac{1-\theta\beta}{2}G_t\pi_t=\kappa\left[\widehat{\xi}_{1t}-\widehat{\xi}_{2t}+\frac{1}{2}(\widehat{\xi}_{1t}^2-\widehat{\xi}_{2t}^2)\right]+\beta\pi_{t+1}$$
$$+\beta\frac{1-\theta\beta}{2}G_{t+1}\pi_{t+1}+\beta\frac{\epsilon-1}{2(1-\theta)}\pi_{t+1}^2+\beta\frac{\epsilon}{2}\pi_{t+1}^2\,. \tag{A.76}$$

Here, the log-linearized Calvo terms are given by $\widehat{\xi}_{1t}\equiv mc_t-\sigma\widehat{c}_t+\widehat{y}_t$ and $\widehat{\xi}_{2t}\equiv\widehat{y}_t-\sigma\widehat{c}_t$ and the auxiliary term G_t is defined as the present value of future Calvo terms

$$G_t\equiv\sum_{\tau=t}^{\infty}(\theta\beta)^{\tau-t}(\widehat{\xi}_{1,t,\tau}+\widehat{\xi}_{2,t,\tau})\,,$$

where $\widehat{\xi}_{1,t,\tau}\equiv\widehat{\xi}_{1\tau}+\epsilon\sum_{s=t+1}^{\tau}\pi_s$ and $\widehat{\xi}_{1,t,\tau}\equiv\xi_{1\tau}+(\epsilon-1)\sum_{s=t+1}^{\tau}\pi_s$. Defining $H_t\equiv\pi_t+\frac{\epsilon-1}{2(1-\theta)}\pi_t^2+\frac{1-\theta\beta}{2}G_t\pi_t+\frac{\epsilon}{2}\pi_t^2$, the extended NKPC (A.76) can be rewritten as:

$$H_t=\kappa\left[\widehat{\xi}_{1t}-\widehat{\xi}_{2t}+\frac{1}{2}(\widehat{\xi}_{1t}^2-\widehat{\xi}_{2t}^2)\right]+\beta\frac{\epsilon}{2}\pi_t^2+\beta H_{t+1}.$$

Iterating forward:

$$H_0=\kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\widehat{\xi}_{1t}-\widehat{\xi}_{2t}+\frac{1}{2}(\widehat{\xi}_{1t}^2-\widehat{\xi}_{2t}^2)\right\}\right]+\frac{\epsilon}{2}\sum_{t=0}^{\infty}\beta^t\pi_t^2\,. \tag{A.77}$$

Now, using market clearing (A.69) and production function (A.63):

$$\begin{aligned}\widehat{\xi}_{1t}=&\widehat{mc}_t+\widehat{y}_t-\sigma\widehat{c}_t=\widehat{w}_t-a_t-\widehat{\Lambda}_t+\widehat{y}_t-\sigma\widehat{y}_t=\varphi\widehat{n}_t-a_t-\widehat{\Lambda}_t+\widehat{y}_t\\=&\varphi(\widehat{\Delta}_t+\widehat{y}_t-a_t-\widehat{\Lambda}_t)-a_t-\widehat{\Lambda}_t+\widehat{y}_t=(1+\varphi)\widehat{y}_t-(1+\varphi)a_t-(1+\varphi)\widehat{\Lambda}_t+\varphi\widehat{\Delta}_t\,,\end{aligned}$$

Hence, the difference between the Calvo terms reduces to:

$$\widehat{\xi}_{1t}-\widehat{\xi}_{2t}\approx(\sigma+\varphi)\widehat{y}_t-(1+\varphi)\widehat{\Lambda}_t+\varphi\widehat{\Delta}_t.$$

The difference between the squared Calvo terms, ignoring terms of order higher than two and terms irrelevant for policy, simplifies to:

$$\begin{aligned}\widehat{\xi}_{1t}^2-\widehat{\xi}_{2t}^2=&[(1+\varphi)\widehat{y}_t-(1+\varphi)a_t-(1+\varphi)\Lambda_t+\varphi\widehat{\Delta}_t]^2-(1-\sigma)^2\widehat{y}_t^2\\\approx&(\sigma+\varphi)(2+\varphi-\sigma)\widehat{y}_t^2+(1+\varphi)^2[\widehat{\Lambda}_t^2-2\widehat{y}_t a_t-2\widehat{y}_t\widehat{\Lambda}_t+2a_t\widehat{\Lambda}_t].\end{aligned}$$

Hence, we can then rewrite the discounted sum H_0 as

$$H_0\approx\kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{(\sigma+\varphi)\widehat{y}_t+\varphi\widehat{\Delta}_t-(1+\varphi)\widehat{\Lambda}_t+\frac{1}{2}(\sigma+\varphi)(2+\varphi-\sigma)\widehat{y}_t^2+\frac{(1+\varphi)^2}{2}\widehat{\Lambda}_t^2-\right.\right.$$
$$\left.\left.-(1+\varphi)^2[\widehat{y}_t a_t+\widehat{y}_t\widehat{\Lambda}_t-a_t\widehat{\Lambda}_t]+\frac{\epsilon}{2\kappa}\pi_t^2\right\}\right].$$

We plug in the second-order approximation of emission damages (A.74):

$$\begin{aligned} H_0 \approx & \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\zeta y_t+\varphi\Delta_t+\frac{1}{2}[(\sigma+\varphi)(2+\varphi-\sigma)+(1+\varphi)\gamma(1-\gamma)]\hat{y}_t^2\right.\right. \\ & \left.\left.+(2+\varphi)\frac{1+\varphi}{2}\hat{\Lambda}_t^2-(1+\varphi)^2[\hat{y}_t a_t+\hat{y}_t\hat{\Lambda}_t-a_t\hat{\Lambda}_t]+\frac{\epsilon}{2\kappa}\pi_t^2\right\}\right]. \end{aligned}$$

Since H_0 is given and the present value of price dispersions is linked to the present value of squared inflation $\sum\beta^t\Delta_t=\frac{\epsilon}{2\kappa}\sum\beta^t\pi_t^2$, we then have:

$$\begin{aligned} \kappa\zeta\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\hat{y}_t\right]\approx & -\frac{\kappa}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2+[(\sigma+\varphi)(2+\varphi-\sigma)+(1+\varphi)\gamma(1-\gamma)]\hat{y}_t^2+\right.\right. \\ & \left.\left.+(2+\varphi)(1+\varphi)\hat{\Lambda}_t^2-2(1+\varphi)^2[\hat{y}_t a_t+\hat{y}_t\hat{\Lambda}_t-a_t\hat{\Lambda}_t]\right\}\right]. \end{aligned}$$

Expressing in terms of the discounted sum of output:

$$\begin{aligned} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\hat{y}_t\right]\approx & \frac{1}{2\zeta}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{-\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2-[(\sigma+\varphi)(2+\varphi-\sigma)+(1+\varphi)\gamma(1-\gamma)]\hat{y}_t^2-\right.\right. \\ & \left.\left.-(2+\varphi)(1+\varphi)\hat{\Lambda}_t^2+2(1+\varphi)^2[\hat{y}_t a_t+\hat{y}_t\hat{\Lambda}_t-a_t\hat{\Lambda}_t]\right\}\right]. \end{aligned}$$

Having expressed the linear terms in the loss function by quadratic terms, we **plug these quadratic terms back** into the loss function that still contains the linear terms (A.75) to arrive at a loss function in terms of second order terms for inflation, output, and emission damages:

$$\mathcal{L}\approx\frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi\pi_t^2+\Omega_y(\hat{y}_t)^2+\Omega_\Lambda\hat{\Lambda}_t^2-2\Omega_{ya}\hat{y}_t a_t-2\Omega_{y\Lambda}\hat{y}_t\hat{\Lambda}_t-2\Omega_{a\Lambda}\hat{\Lambda}_t a_t\right\}\right], \tag{A.78}$$

with auxiliary parameters

$$\begin{aligned} \Omega_\pi &= \frac{\epsilon}{\kappa}\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right] \\ \Omega_y &= -1+\sigma+\frac{1+\Phi}{1+\gamma}[1+\varphi+\gamma(1-\gamma)]-\Phi\frac{(\sigma+\varphi)(2+\varphi-\sigma)+(1+\varphi)\gamma(1-\gamma)}{\zeta} \\ &= \gamma(1-\gamma)\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right]-1+\sigma+\frac{1+\Phi}{1+\gamma}(1+\varphi)-\frac{\Phi}{\zeta}[(1+\varphi)-(1-\sigma)][1+\varphi+1-\sigma] \\ &= \gamma(1-\gamma)\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right]-1+\sigma+\frac{1+\Phi}{1+\gamma}(1+\varphi)-\frac{\Phi}{\zeta}[(1+\varphi)^2-(1-\sigma)^2] \\ &= [\gamma(1-\gamma)+1+\varphi]\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right]-(1-\sigma)\left(1-\frac{\Phi}{\zeta}(1-\sigma)\right) \\ &= [\gamma(1-\gamma)+1+\varphi]\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right]-(1-\sigma)\left(1-\frac{\Phi}{\zeta}(1+\varphi+\gamma(1+\varphi)-\zeta)\right) \\ &= [\gamma(1-\gamma)+1+\varphi-(1-\sigma)(1+\gamma)]\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right] \\ \Omega_\Lambda &= (2+\varphi)\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right] \\ \Omega_{ya} &= (1+\varphi)\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right] \\ \Omega_{y\Lambda} &= (1+\varphi)\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right] \\ \Omega_{a\Lambda} &= -(1+\varphi)\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right] \end{aligned}$$

Eliminating the linear terms in (A.78) using $\hat{\Lambda}_t=-\gamma\hat{y}_t-\frac{1}{2}\gamma(1-\gamma)\hat{y}_t^2-\frac{1}{2}\hat{\Lambda}_t^2$, we get:

$$\mathcal{L}\approx\frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi\pi_t^2+\tilde{\Omega}_y\hat{y}_t^2-2\tilde{\Omega}_{ya}\hat{y}_t a_t\right\}\right], \tag{A.79}$$

with two composite auxiliary parameters associated with the squared output response $\hat{y}_t^2$ and the interaction term $\hat{y}_t a_t$:

$$\begin{aligned} \tilde{\Omega}_y &= \Omega_y+\gamma^2\cdot\Omega_\Lambda+2\gamma\Omega_{y\Lambda}=\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right][\zeta(1+\gamma)+\gamma], \\ \tilde{\Omega}_{ya} &= \Omega_{ya}-\gamma\Omega_{a\Lambda}=\left[\frac{1+\Phi}{1+\gamma}-\Phi\frac{1+\varphi}{\zeta}\right](1+\varphi)(1+\gamma). \end{aligned}$$

To ease interpretation, we substitute the definition of the natural output gap $\hat{y}_t = x_t^n + \hat{y}_t^n = x_t^n + \frac{1+\varphi}{\zeta} a_t$:

$$\begin{aligned}\mathcal{L} &\approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi \pi_t^2 + \Omega_x (x_t^n)^2 - 2\Omega_{xa} x_t^n a_t\right\}\right] \\ &= \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi \pi_t^2 + \Omega_x\left((x_t^n)^2 - 2\frac{\Omega_{xa}}{\Omega_x} x_t^n a_t\right)\right\}\right], \end{aligned} \tag{A.80}$$

with auxiliary parameters

$$\Omega_x = \widetilde{\Omega}_y \quad \text{and} \quad \Omega_{xa} = \widetilde{\Omega}_{ya} - \frac{1+\varphi}{\zeta}\widetilde{\Omega}_y.$$

We can exploit that the expression $\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right]$ cancels out when normalizing the weight on inflation in the loss function to one. Adding and subtracting $\frac{\Omega_{xa}^2}{\Omega_x^2}$ delivers an expression for the loss function in terms of squared inflation and natural output deviations from a target level x_t^*:

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\pi_t^2 + \omega_x\left(x_t - x_t^*\right)^2\right\}\right], \tag{A.81}$$

with $\omega_x = \frac{\Omega_x}{\Omega_\pi} = \frac{\kappa}{\epsilon}(\zeta(1+\gamma)+\gamma)$ and $x_t^* = \frac{\Omega_{xa}}{\Omega_x} a_t = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma} a_t$. □

A.6. Proof of Proposition 5

Under discretion, the central bank takes expected inflation and output gap as given when choosing inflation and natural output gap to minimize

$$\begin{aligned}&\min_{\pi_t, x_t^n} \quad \frac{1}{2}\mathbb{E}_0\left[\pi_t^2 + \omega_x (x_t^n - x_t^*)^2\right] \\ &\text{s.t.} \quad \pi_t = \zeta\kappa x_t^n + \beta\mathbb{E}_t[\pi_{t+1}]. \end{aligned} \tag{A.82}$$

Denoting the multiplier on the constraint by λ_t, the first-order conditions are given by

$$-\omega_x(x_t^n - x_t^*) - \lambda_t\zeta\kappa = 0 \qquad \text{and} \qquad \lambda_t = \pi_t .$$

Combining them yields the optimal monetary policy condition (24) that summarizes the trade-off between the natural output gap x_t^n and inflation π_t. If productivity shocks are i.i.d., we have $\mathbb{E}_t[\pi_{t+1}] = \mathbb{E}_t[x_{t+1}^n] = 0$. Plugging the monetary policy rule (24) into the NKPC (A.82), we can re-arrange for the optimal output gap reaction x_t^o in Eq. (26). Plugging the optimal output gap reaction into the NKPC, we get to Eq. (25). Plugging both inside the IS curve (16) and solving for the optimal monetary policy rate r_t^o, we get to equation Eq. (27). □

Appendix B. Proofs for demand and cost-push shocks

In this Appendix, we provide further details on the derivations of the central bank loss function and optimal monetary policy in the presence of demand and cost-push shocks.

B.1. Proof of Proposition 6

The presence of demand and cost-push shocks does not affect the approximation of the discounted lifetime utility derived in Appendix A.5 until (A.75). A second order approximation of Eq. (A.53) in the presence of cost-push shocks leads to the following (extended) NKPC:

$$\begin{aligned}\pi_t + \frac{\epsilon-1}{2(1-\theta)}\pi_t^2 + \frac{1-\theta\beta}{2}G_t\pi_t =& \kappa\left[\hat{\xi}_{1t} - \hat{\xi}_{2t} + \hat{\mu}_t + \frac{1}{2}(\hat{\xi}_{1t}^2 - \hat{\xi}_{2t}^2)\right] + \beta\pi_{t+1} \\ &+ \beta\frac{1-\theta\beta}{2}G_{t+1}\pi_{t+1} + \beta\frac{\epsilon-1}{2(1-\theta)}\pi_{t+1}^2 + \beta\frac{\epsilon}{2}\pi_{t+1}^2 . \end{aligned} \tag{B.83}$$

Here, the log-linearized Calvo terms are given by $\hat{\xi}_{1t} \equiv mc_t - \sigma\hat{c}_t + \hat{y}_t$ and $\hat{\xi}_{2t} \equiv \hat{y}_t - \sigma\hat{c}_t$ and the auxiliary term G_t is defined as the present value of future Calvo terms

$$G_t \equiv \sum_{\tau=t}^{\infty}(\theta\beta)^{\tau-t}(\hat{\xi}_{1,t,\tau} + \hat{\xi}_{2,t,\tau}),$$

where $\hat{\xi}_{1,t,\tau} \equiv \hat{\xi}_{1\tau} + \epsilon\sum_{s=t+1}^{\tau}\pi_s$ and $\hat{\xi}_{1,t,\tau} \equiv \xi_{1\tau} + (\epsilon-1)\sum_{s=t+1}^{\tau}\pi_s$. Defining $H_t \equiv \pi_t + \frac{\epsilon-1}{2(1-\theta)}\pi_t^2 + \frac{1-\theta\beta}{2}G_t\pi_t + \frac{\epsilon}{2}\pi_t^2$, the extended NKPC (B.83) can be rewritten as:

$$H_t = \kappa\left[\hat{\xi}_{1t} - \hat{\xi}_{2t} + \hat{\mu}_t + \frac{1}{2}(\hat{\xi}_{1t}^2 - \hat{\xi}_{2t}^2)\right] + \beta\frac{\epsilon}{2}\pi_t^2 + \beta H_{t+1}.$$

Iterating forward:

$$H_0 = \kappa \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \hat{\xi}_{1t} - \hat{\xi}_{2t} + \hat{\mu}_t + \frac{1}{2}(\hat{\xi}_{1t}^2 - \hat{\xi}_{2t}^2) \right\} \right] + \frac{\epsilon}{2} \sum_{t=0}^{\infty} \beta^t \pi_t^2 \,. \tag{B.84}$$

Now, using market clearing (A.69) and production function (A.63):

$$\begin{aligned}\hat{\xi}_{1t} =& \widehat{mc}_t + \hat{y}_t - \sigma \hat{c}_t = \hat{w}_t - a_t - \hat{\Lambda}_t + \hat{y}_t - \sigma \hat{y}_t = \varphi \hat{n}_t - a_t - \hat{\Lambda}_t + \hat{y}_t \\ =& \varphi(\hat{\Delta}_t + \hat{y}_t - a_t - \hat{\Lambda}_t) - a_t - \hat{\Lambda}_t + \hat{y}_t = (1+\varphi)\hat{y}_t - (1+\varphi)a_t - (1+\varphi)\hat{\Lambda}_t + \varphi\hat{\Delta}_t \,,\end{aligned}$$

Hence, the difference between the Calvo terms reduces to:

$$\hat{\xi}_{1t} - \hat{\xi}_{2t} \approx (\sigma+\varphi)\hat{y}_t - (1+\varphi)\hat{\Lambda}_t + \varphi\hat{\Delta}_t.$$

The difference between the squared Calvo terms, ignoring terms of order higher than two and terms irrelevant for policy, simplifies to:

$$\begin{aligned}\hat{\xi}_{1t}^2 - \hat{\xi}_{2t}^2 =& [(1+\varphi)\hat{y}_t - (1+\varphi)a_t - (1+\varphi)\Lambda_t + \varphi\hat{\Delta}_t]^2 - (1-\sigma)^2\hat{y}_t^2 \\ \approx& (\sigma+\varphi)(2+\varphi-\sigma)\hat{y}_t^2 + (1+\varphi)^2[\hat{\Lambda}_t^2 - 2\hat{y}_t a_t - 2\hat{y}_t\hat{\Lambda}_t + 2a_t\hat{\Lambda}_t].\end{aligned}$$

Hence, we can then rewrite the discounted sum H_0 as

$$\begin{aligned}H_0 \approx& \kappa \mathbb{E}_0 \Bigg[\sum_{t=0}^{\infty} \beta^t \Bigg\{ (\sigma+\varphi)\hat{y}_t + \varphi\hat{\Delta}_t - (1+\varphi)\hat{\Lambda}_t + \hat{\mu}_t + \frac{1}{2}(\sigma+\varphi)(2+\varphi-\sigma)\hat{y}_t^2 + \frac{(1+\varphi)^2}{2}\hat{\Lambda}_t^2 - \\ &- (1+\varphi)^2[\hat{y}_t a_t + \hat{y}_t\hat{\Lambda}_t - a_t\hat{\Lambda}_t] + \frac{\epsilon}{2\kappa}\pi_t^2 \Bigg\} \Bigg].\end{aligned}$$

We plug in the second-order approximation of emission damages (A.74):

$$\begin{aligned}H_0 \approx& \kappa \mathbb{E}_0 \Bigg[\sum_{t=0}^{\infty} \beta^t \Bigg\{ \zeta y_t + \varphi\Delta_t + \hat{\mu}_t + \frac{1}{2}[(\sigma+\varphi)(2+\varphi-\sigma) + (1+\varphi)\gamma(1-\gamma)]\hat{y}_t^2 \\ &+ (2+\varphi)\frac{1+\varphi}{2}\hat{\Lambda}_t^2 - (1+\varphi)^2[\hat{y}_t a_t + \hat{y}_t\hat{\Lambda}_t - a_t\hat{\Lambda}_t] + \frac{\epsilon}{2\kappa}\pi_t^2 \Bigg\} \Bigg].\end{aligned}$$

Since H_0 is given and the present value of price dispersions is linked to the present value of squared inflation $\sum \beta^t \Delta_t = \frac{\epsilon}{2\kappa}\sum \beta^t \pi_t^2$, we then have:

$$\begin{aligned}\kappa\zeta\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\hat{y}_t\right] \approx& -\frac{\kappa}{2}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 + [(\sigma+\varphi)(2+\varphi-\sigma) + (1+\varphi)\gamma(1-\gamma)]\hat{y}_t^2 + \\ &+ (2+\varphi)(1+\varphi)\hat{\Lambda}_t^2 - 2(1+\varphi)^2[\hat{y}_t a_t + \hat{y}_t\hat{\Lambda}_t - a_t\hat{\Lambda}_t] + 2\hat{\mu}_t\Bigg\}\Bigg].\end{aligned}$$

Expressing in terms of the discounted sum of output:

$$\begin{aligned}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\hat{y}_t\right] \approx& \frac{1}{2\zeta}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{-\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 - [(\sigma+\varphi)(2+\varphi-\sigma) + (1+\varphi)\gamma(1-\gamma)]\hat{y}_t^2 - \\ &- (2+\varphi)(1+\varphi)\hat{\Lambda}_t^2 + 2(1+\varphi)^2[\hat{y}_t a_t + \hat{y}_t\hat{\Lambda}_t - a_t\hat{\Lambda}_t] - 2\hat{\mu}_t\Bigg\}\Bigg].\end{aligned}$$

Having expressed the linear terms in the loss function by quadratic terms, notice now that the cost-push shock enters in isolation and becomes a term independent of policy when we **plug these quadratic terms back** into the loss function that still contains the linear terms (A.75), which also arose in the case of TFP shocks. We then arrive at the same loss function as in (A.79):

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi\pi_t^2 + \widetilde{\Omega}_y\hat{y}_t^2 - 2\widetilde{\Omega}_{ya}\hat{y}_t a_t\right\}\right] \,,$$

with two composite auxiliary parameters associated with the squared output response $\hat{y}_t^2$ and the interaction term $\hat{y}_t a_t$:

$$\begin{aligned}\widetilde{\Omega}_y &= \Omega_y + \gamma^2 \cdot \Omega_\Lambda + 2\gamma\Omega_{y\Lambda} = \left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right][\zeta(1+\gamma)+\gamma] \,, \\ \widetilde{\Omega}_{ya} &= \Omega_{ya} - \gamma\Omega_{a\Lambda} = \left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right](1+\varphi)(1+\gamma) \,.\end{aligned}$$

Now, it is trivial to show that the demand shock has no effects neither on the efficient level nor on the natural level, while the cost-push shock only affects the natural level of output (in the case of constant carbon tax) in the following way:

$$y_t^n = \frac{1+\varphi}{\zeta}a_t - \frac{1}{\zeta}\mu_t. \tag{B.85}$$

Hence, when we substitute the definition of the natural output gap $\hat{y}_t = x_t^n + \hat{y}_t^n = x_t^n + \frac{1+\varphi}{\zeta} a_t - \frac{1}{\zeta}\mu_t$, we get:

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \Omega_\pi \pi_t^2 + \Omega_x \left((x_t^n)^2 - 2x_t^n \left[\frac{\Omega_{xa}}{\Omega_x} a_t + \frac{\widetilde{\Omega}_y}{\Omega_x \zeta} \mu_t \right] \right) \right\} \right], \tag{B.86}$$

with auxiliary parameters

$$\Omega_x = \widetilde{\Omega}_y \quad \text{and} \quad \Omega_{xa} = \widetilde{\Omega}_{ya} - \frac{1+\varphi}{\zeta}\widetilde{\Omega}_y.$$

We can exploit that the expression $\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right]$ cancels out when normalizing the weight on inflation in the loss function to one. Adding and subtracting $\frac{\Omega_{xa}^2}{\Omega_x^2} a_t^2 + \frac{\widetilde{\Omega}_y^2}{\Omega_x^2 \zeta^2}\mu_t^2$ delivers an expression for the loss function in terms of squared inflation and natural output deviations from a target level x_t^*:

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \pi_t^2 + \omega_x \left(x_t - x_t^*\right)^2 \right\} \right], \tag{B.87}$$

with $\omega_x = \frac{\Omega_x}{\Omega_\pi} = \frac{\kappa}{\epsilon}(\zeta(1+\gamma)+\gamma)$ and $x_t^* = \frac{\Omega_{xa}}{\Omega_x} a_t + \frac{\widetilde{\Omega}_y}{\Omega_x \zeta}\mu_t = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma} a_t + \frac{1}{\zeta}\mu_t$. □

B.2. Proof of Proposition 7

Under discretion, the central bank takes expected inflation and output gap as given when choosing current inflation and natural output gap to minimize

$$\min_{\pi_t, x_t^n} \quad \frac{1}{2}\mathbb{E}_0 \left[\pi_t^2 + \omega_x (x_t^n - x_t^*)^2 \right]$$
$$\text{s.t.} \quad \pi_t = \zeta\kappa x_t^n + \beta \mathbb{E}_t[\pi_{t+1}] + \mu_t \,. \tag{B.88}$$

If shocks are i.i.d., we have $\mathbb{E}_t[\pi_{t+1}] = \mathbb{E}_t[x_{t+1}^n] = 0$. Optimal responses of x_t^n and π_t then solve the system composed of the monetary policy rule (24) and the NKPC (B.88). Since ρ_t does not enter neither of the equations, it follows that $x_t^n = \pi_t = 0$ is a solution of the central bank's problem in the presence of a demand shock. Plugging $x_t^n = \pi_t = E_t[\pi_{t+1}] = E_t[x_{t+1}] = 0$ into (29), we get $r_t^o = \frac{1}{\sigma}\mu_t = r_t^n$.

Considering instead the response to a cost-push shock, μ_t, plugging the monetary policy rule (24) into the NKPC (B.88), we can re-arrange for the optimal output reaction:

$$x_t^o = \frac{\omega_x}{\zeta^2\kappa^2 + \omega_x} x_t^* - \mu_t = \frac{\omega_x - \zeta^2\kappa}{\zeta(\omega_x + \zeta^2\kappa^2)}\mu_t. \tag{B.89}$$

Plugging this into the NKPC (B.88), we get (37). Plugging both conditions into the IS curve (29) and solving for the optimal monetary policy rate r_t^o, we get Eq. (39). □

Appendix C. On the central bank loss function

In this section, we derive the central bank loss function that does not take into account that output expansions affect economic damages from emissions. In this case, approximating household utility and the NKPC up to second order and combining it with the production function and market clearing condition (Benigno and Woodford, 2005) yields a loss function that inherits the inefficiency of the flexible price equilibrium. Consequently, this loss function prescribes to close natural output gap and inflation gap to zero in all states, which is feasible in our model. Notably, this loss function takes the presence of climate change into account in so far as it affects the natural rate of interest, i.e. it is *climate conscious* (Nakov and Thomas, 2023). However, such a loss function is not consistent with an utilitarian welfare criterion.

The **second order approximation** of the welfare objective, i.e. households period utility function U_t, remains unchanged:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \hat{c}_t + \frac{\hat{c}_t^2}{2} - \frac{\sigma}{2}\hat{c}_t^2 - (1-\tau^c)\left[\hat{n}_t + \frac{\hat{n}_t^2}{2} + \frac{\varphi}{2}\hat{n}_t^2\right].$$

As in the main text, in order to express the loss function in terms of the natural output gap x_t^n and inflation π_t, we make use of the market clearing condition $\hat{c}_t = \hat{y}_t$. Different to before, we directly exploit (A.62) to re-write the production function as $\hat{n}_t = \hat{y}_t + \hat{\Delta}_t - a_t - \hat{\Lambda}_t = (1+\gamma)\hat{y}_t + \hat{\Delta}_t - a_t$:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \hat{y}_t + \frac{1-\sigma}{2}\hat{y}_t^2 - (1-\tau)(1+\gamma)\left[\hat{y}_t + \frac{\hat{\Delta}_t}{1+\gamma} - \frac{a_t}{1+\gamma} + \frac{1+\varphi}{2(1+\gamma)}\left((1+\gamma)\hat{y}_t + \hat{\Delta}_t - a_t\right)^2\right],$$

such that $\hat{\Lambda}_t$ does not enter the objective function. Eliminating all terms independent of policy and of order higher than two, we then obtain:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \hat{y}_t + \frac{1-\sigma}{2}\hat{y}_t^2 - (1+\Phi)\left[\left(\hat{y}_t + \hat{\Delta}_t\frac{1}{1+\gamma} + \frac{1+\varphi}{2}(1+\gamma)\hat{y}_t^2 - (1+\varphi)y_t a_t\right)^2\right] + t.i.p.$$

$$\approx -\Phi\hat{y}_t - \frac{\hat{y}_t^2}{2}\Big[-1+\sigma+(1+\Phi)(1+\varphi)(1+\gamma)\Big] - \frac{1+\Phi}{1+\gamma}\hat{\Delta}_t + (1+\Phi)(1+\varphi)\hat{y}_t a_t + t.i.p.$$
$$\approx -\Phi\hat{y}_t - \frac{\hat{y}_t^2}{2}\Big[\zeta(1+\Phi)+\Phi(1-\sigma)\Big] - \frac{1+\Phi}{1+\gamma}\hat{\Delta}_t + (1+\Phi)(1+\varphi)\hat{y}_t a_t + t.i.p.$$

Using the definition of the natural output level (13) and of the natural output gap $\hat{y}_t = x_t^n + \hat{y}_t^n = x_t^n + \frac{1+\varphi}{\zeta}a_t$ and eliminating all terms independent of policy, we arrive at:

$$\frac{U_t - U}{c^{1-\sigma}} \approx -\Phi x_t^n - \frac{\zeta(1+\Phi)+\Phi(1-\sigma)}{2}\Big((x_t^n)^2 + 2\frac{1+\varphi}{\zeta}x_t^n a_t\Big) - \frac{1+\Phi}{1+\gamma}\hat{\Delta}_t +$$
$$+ (1+\Phi)(1+\varphi)x_t^n a_t + t.i.p.$$
$$\approx -\Phi x_t^n - \frac{1}{2}\Big\{\zeta(1+\Phi)+\Phi(1-\sigma)\Big\}(x_t^n)^2 - \frac{1+\Phi}{1+\gamma\widetilde{\Phi}}\hat{\Delta}_t - \frac{\Phi}{\zeta}(1-\sigma)(1+\varphi)x_t^n a_t + t.i.p.$$

We are then ready to **evaluate the loss function**:

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(\zeta(1+\Phi)+\Phi(1-\sigma)\Big)(x_t^n)^2 + \frac{1+\Phi}{1+\gamma}\hat{\Delta}_t + \right.\right.$$
$$\left.\left. + \Phi\frac{(1+\varphi)}{\zeta}(1-\sigma)x_t^n a_t + \Phi x_t^n\right\}\right].$$

The discounted sum of log price dispersions is given by $\sum_{t=0}^{\infty}\beta^t\hat{\Delta}_t \approx \frac{\epsilon}{2\kappa}\sum_{t=0}^{\infty}\beta^t\pi_t^2$, with the auxiliary parameter $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta}$ governing the slope of the NKPC. Therefore, the loss function is given by

$$\mathcal{L} \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(\zeta(1+\Phi)+\Phi(1-\sigma)\Big)(x_t^n)^2 + \frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\hat{\Delta}_t\right.\right.$$
$$\left.\left. + \Phi\frac{(1+\varphi)}{\zeta}(1-\sigma)x_t^n a_t + \Phi x_t^n\right\}\right]. \tag{C.90}$$

Eliminating Linear Terms. There is only one linear term Φx_t^n in this expression and we exploit that the same (extended) NKPC (A.76), which can be iterated forward to:

$$H_0 = \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\hat{\xi}_{1t} - \hat{\xi}_{2t} + \frac{1}{2}(\hat{\xi}_{1t}^2 - \hat{\xi}_{2t}^2)\right\}\right] + \frac{\epsilon}{2}\sum_{t=0}^{\infty}\beta^t\pi_t^2 . \tag{C.91}$$

Now, using market clearing (A.69) and the re-written production function $\hat{n}_t = (1+\gamma)\hat{y}_t + \hat{\Delta}_t - a_t$, we can write the Calvo terms without taking the terms related to $\hat{\Lambda}_t$ into account:

$$\hat{\xi}_{1t} = \widehat{mc}_t + \hat{y}_t - \sigma\hat{c}_t = \hat{w}_t - a_t + \gamma\hat{y}_t + \hat{y}_t - \sigma\hat{y}_t = \varphi\hat{n}_t - a_t + (1+\gamma)\hat{y}_t$$
$$= \varphi(\hat{\Delta}_t + \hat{y}_t - a_t + \gamma\hat{y}_t) - a_t + (1+\gamma)\hat{y}_t = (1+\zeta-\sigma)\hat{y}_t - (1+\varphi)a_t + \varphi\hat{\Delta}_t ,$$

Hence, the difference between the Calvo terms reduces to:

$$\hat{\xi}_{1t} - \hat{\xi}_{2t} = (1+\zeta)\hat{y}_t - (1+\varphi)a_t + \varphi\hat{\Delta}_t - \hat{y}_t = \zeta\hat{y}_t - (1+\varphi)a_t + \varphi\hat{\Delta}_t \approx \zeta\hat{y}_t + \varphi\hat{\Delta}_t.$$

The difference between the squared Calvo terms, ignoring terms of order higher than two and terms irrelevant for policy, simplifies to:

$$\hat{\xi}_{1t}^2 - \hat{\xi}_{2t}^2 = [(1+\zeta-\sigma)\hat{y}_t - (1+\varphi)a_t + \varphi\hat{\Delta}_t]^2 - (1-\sigma)^2\hat{y}_t^2$$
$$= [(1+\zeta-\sigma)^2 - 1]\hat{y}_t^2 - 2(1+\zeta-\sigma)(1+\varphi)\hat{y}_t a_t$$
$$= \zeta(\zeta+2-2\sigma)\hat{y}_t^2 - 2(1+\zeta-\sigma)(1+\varphi)\hat{y}_t a_t.$$

Hence, we can then rewrite the discounted sum H_0 as

$$H_0 \approx \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\zeta y_t + \varphi\Delta_t + \frac{1}{2}\zeta(\zeta+2-2\sigma)\hat{y}_t^2 - (1+\zeta-\sigma)(1+\varphi)\hat{y}_t a_t + \frac{\epsilon}{2\kappa}\pi_t^2\right\}\right]. \tag{C.92}$$

Since H_0 is given and the price dispersion terms can be expressed in terms of inflation $\sum\beta^t\Delta_t = \frac{\epsilon}{2\kappa}\sum\beta^t\pi_t^2$, we then can rearrange (C.92) for the discounted sum of output deviations:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\hat{y}_t\right] \approx \frac{1}{2\zeta}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{-\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 - \zeta(\zeta+2-2\sigma)\hat{y}_t^2 + 2(1+\zeta-\sigma)(1+\varphi)\hat{y}_t a_t\right\}\right].$$

Rewriting in terms of the natural output gap $\hat{y}_t = x_t^n + \hat{y}_t^n = x_t^n + \frac{1+\varphi}{\zeta}a_t$, and ignoring terms of order higher than two and irrelevant for policy, we get:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t x_t^n\right] \approx \frac{\mathbb{E}_0}{2\zeta}\left[\sum_{t=0}^{\infty}\beta^t\left\{-\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 - \zeta(\zeta+2-2\sigma)\Big((x_t^n)^2 + 2\frac{1+\varphi}{\zeta}x_t^n a_t\Big)\right.\right.$$

$$+ 2(1+\zeta-\sigma)(1+\varphi)x_t^n a_t \Bigg\}\Bigg],$$

which implies that we can express the present value of natural output gaps as

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t x_t^n\right] \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{1}{2}X_1\pi_t^2 + \frac{1}{2}X_2(x_t^n)^2 + X_3 x_t^n a_t\right)\right], \tag{C.93}$$

where the auxiliary terms are defined as

$$X_1 = -\frac{\epsilon}{\kappa}\frac{1+\varphi}{\zeta},\ X_2 = -(\zeta+2-2\sigma),\ X_3 = -(1+\varphi)\frac{1-\sigma}{\zeta}.$$

Having expressed the linear terms in the loss function by quadratic terms, we **plug these quadratic terms back** into (C.90):

$$\begin{aligned}
\mathcal{L} &\approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(\zeta+\Phi(\zeta+1-\sigma)\Big)(x_t^n)^2 + \frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\pi_t^2\right.\right. \\
&\qquad \left.\left. + \Phi\frac{1+\varphi}{\zeta}(1-\sigma)x_t^n a_t + \Phi x_t^n\right\}\right] \\
&\approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(\zeta+\Phi(\zeta+1-\sigma)+\Phi X_2\Big)(x_t^n)^2 + \frac{1}{2}\left(\frac{1+\Phi}{1+\gamma}\frac{\epsilon}{\kappa}+\Phi X_1\right)\pi_t^2\right.\right. \\
&\qquad \left.\left. + \left(\frac{1+\varphi}{\zeta}\Big[\Phi(1-\sigma)\Big] + \Phi X_3\right)x_t^n a_t\right\}\right] \\
&\approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\frac{\epsilon}{\kappa}\frac{\zeta-\Phi(1-\sigma)}{\zeta(1+\gamma)}\pi_t^2 + \frac{1}{2}\Big(\zeta-\Phi(1-\sigma)\Big)(x_t^n)^2\right\}\right] \\
&\approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi\pi_t^2 + \Omega_x(x_t^n)^2\right\}\right],
\end{aligned} \tag{C.94}$$

with auxiliary parameters

$$\Omega_\pi = \frac{\epsilon}{\kappa}\frac{\zeta-\Phi(1-\sigma)}{\zeta(1+\gamma)},\ \Omega_x = \zeta-\Phi(1-\sigma).$$

As a last step, we get:

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\pi_t^2 + \omega_x\left(x_t^n\right)^2\right\}\right], \tag{C.95}$$

where $\omega_x = \frac{\Omega_x}{\Omega_\pi} = \frac{\kappa}{\epsilon}\zeta(1+\gamma)$. □

Different to the loss function in the main text, (C.95) does not include a target level for the natural output gap. While the weight on output stabilization is still positively related to the externality, divine coincidence still holds: a central bank that chooses not to internalize the effect of output expansions on economic damages associated with emission externalities can and should perfectly close inflation and natural output gap.

Appendix D. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.euroecorev.2025.105124.

Data availability

All data replication files are available as supplementary online material.

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